

On the generalized resolvents of isometric operators with gaps.

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1 Introduction.

We shall investigate generalized resolvents of an isometric operator. Let V be a closed isometric operator in a (separable) Hilbert space H . There always exists (at least one) unitary operator $U \supseteq V$ in a Hilbert space $\tilde{H} \supseteq H$. Recall that the following operator-valued function \mathbf{R}_ζ :

$$\mathbf{R}_\zeta h = P_H^{\tilde{H}} (E_{\tilde{H}} - \zeta U)^{-1} h, \quad h \in H,$$

is said to be the **generalized resolvent** of the isometric operator V (corresponding to the extension U). Let $\{F_t\}_{t \in [0, 2\pi]}$ be the left-continuous orthogonal resolution of unity of U . Then the following operator-valued function:

$$\mathbf{F}_t = P_H^{\tilde{H}} F_t, \quad t \in [0, 2\pi],$$

is said to be a (left-continuous) **spectral function** of the isometric operator V (corresponding to the extension U). Let $F(\delta)$, $\delta \in \mathfrak{B}(\mathbb{T})$, be the orthogonal spectral measure of U . Then

$$\mathbf{F}(\delta) = P_H^{\tilde{H}} F(\delta), \quad \delta \in \mathfrak{B}(\mathbb{T}),$$

is said to be a **spectral measure** of the isometric operator V (corresponding to the extension U). Of course, we have

$$\mathbf{F}(\delta_t) = \mathbf{F}_t, \quad \delta_t = \{z = e^{i\varphi} : 0 \leq \varphi < t\}, \quad t \in [0, 2\pi],$$

what follows from the analogous property of the orthogonal measures. We notice that there exists a one-to-one correspondence between spectral functions (spectral measures) and generalized resolvents:

$$(\mathbf{R}_\zeta h, g)_H = \int_{\mathbb{T}} \frac{1}{1 - z\zeta} d(\mathbf{F}(\cdot)h, g)_H = \int_0^{2\pi} \frac{1}{1 - ze^{it}} d(\mathbf{F}_t h, g)_H, \quad \forall h, g \in H, \quad (1)$$

according to the inversion formula [1, p.50].

Let H_1 and H_2 be two arbitrary subspaces of the Hilbert space H . By $\mathcal{S}(H_1; H_2)$ we denote the set of all analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

operator-valued functions $F(\zeta)$ which values are linear contractions with the domain $D(F(\zeta)) = H_1$ and with the range $R(F(\zeta)) \subseteq H_2$, $\forall \zeta \in \mathbb{D}$. Chumakin's formula [2, Theorem 3]:

$$\mathbf{R}_\zeta = [E_H - \zeta(V \oplus F(\zeta))]^{-1}, \quad \zeta \in \mathbb{D}, \quad (2)$$

establishes a one-to-one correspondence between all generalized resolvents of V and all functions $F(\zeta)$ from the set $\mathcal{S}(H \ominus D(V); H \ominus R(V))$.

Set

$$M_\zeta = M_\zeta(V) = (E_H - \zeta V)D(V), \quad N_\zeta = N_\zeta(V) = H \ominus M_\zeta, \quad \zeta \in \mathbb{C};$$

$$M_\infty = M_\infty(V) = R(V), \quad N_\infty = N_\infty(V) = H \ominus R(V).$$

Consider the following operator

$$V_z = (V - \bar{z}E_H)(E_H - zV)^{-1}, \quad z \in \mathbb{D}. \quad (3)$$

Notice that $D(V_z) = M_z$ and $R(V_z) = M_{\frac{1}{\bar{z}}}$. It is straightforward to check that V_z is isometric and

$$V = (V_z + \bar{z}E_H)(E_H + zV_z)^{-1} = (V_z)_{-z}. \quad (4)$$

Moreover, if V is unitary, then V_z is unitary, and vice versa (by (4)).

Let $\widehat{V}_z \supseteq V_z$ be a unitary operator in a Hilbert space $\widehat{H} \supseteq H$. Then we may define the operator

$$\widehat{V} = (\widehat{V}_z + \bar{z}E_{\widehat{H}})(E_{\widehat{H}} + z\widehat{V}_z)^{-1}, \quad (5)$$

which is a unitary extension of V . Formula (5) establishes a one-to-one correspondence between all unitary extensions \widehat{V}_z of V_z in a Hilbert space $\widehat{H} \supseteq H$, and all unitary extensions \widehat{V} of V in a Hilbert space \widehat{H} .

Let us fix an arbitrary point $z_0 \in \mathbb{D}$. Let C be an arbitrary linear bounded operator with the domain $D(C) = N_{z_0}$ and the range $R(C) \subseteq N_{\frac{1}{\bar{z}_0}}$. Set

$$V_{z_0;C}^+ = V_{z_0} \oplus C; \quad (6)$$

$$V_C = V_{C;z_0} = (V_{z_0;C}^+ + \bar{z}_0 E_H)(E_H + z_0 V_{z_0;C}^+)^{-1}. \quad (7)$$

If $z_0 \neq 0$, we may write:

$$V_C = V_{C;z_0} = \frac{1}{z_0} E_H + \frac{|z_0|^2 - 1}{z_0} (E_H + z_0 V_{z_0;C}^+)^{-1}; \quad (8)$$

$$V_{z_0;C}^+ = -\frac{1}{z_0}E_H + \frac{1-|z_0|^2}{z_0}(E_H - z_0V_{C;z_0})^{-1}. \quad (9)$$

Recall that the operator V_C is said to be an **orthogonal extension** of V defined by the operator C .

Inin's formula [3, Theorem]:

$$\mathbf{R}_\zeta = [E - \zeta V_{C(\zeta;z_0)}]^{-1}, \quad \zeta \in \mathbb{D}, \quad (10)$$

establishes a one-to-one correspondence between all generalized resolvents of V and all functions $C(\zeta) = C(\zeta; z_0)$ from the set $\mathcal{S}(N_{z_0}; N_{\frac{1}{\overline{z_0}}})$. Observe that in the case $z_0 = 0$ it coincides with Chumakin's formula.

We shall show that Inin's formula can be derived directly from Chumakin's formula. Then we shall obtain an analog of some McKelvey's results [4, Theorem 2.1 (A),(B); Theorem 3.1 (A),(B); Remark 2.2], see also [5]. Also we obtain an auxiliary proposition which uses some constructions of L.A. Shtraus in [6, Lemma]. All that will be used to obtain some slight correction and generalization of Ryabtseva's results about generalized resolvents of an isometric operator with a gap in [7]. Here we used some ideas of Varlamova-Luks for the case of Hermitian operators with a gap [5],[8],[9].

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\}$, $\mathbb{T}_e = \{z \in \mathbb{C} : |z| \neq 1\}$. By $\mathfrak{B}(\mathbb{T})$ we denote the set of all Borel subsets of \mathbb{T} .

All Hilbert spaces in this paper are assumed to be separable. If H is a Hilbert space then $(\cdot, \cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H , respectively. Indices may be omitted in obvious cases. For a linear operator A in H , we denote by $D(A)$ its domain, by $R(A)$ its range, by $\text{Ker } A$ its null subspace (kernel), and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. \overline{A} means the closure of the operator, if the operator is closable. If A is bounded then $\|A\|$ denotes its norm. The set of all points of the regular type of A is denoted by $\mathcal{M}_r(A)$. For a set $M \subseteq H$ we denote by \overline{M} the closure of M in the norm of H . By $A|_M$ we denote the restriction of the operator A to M . For an arbitrary set of elements $\{x_n\}_{n \in I}$ in H , we denote by $\text{Lin}\{x_n\}_{n \in I}$ the set of all linear combinations of elements x_n , and $\text{span}\{x_n\}_{n \in I} := \overline{\text{Lin}\{x_n\}_{n \in I}}$. Here I is an arbitrary set of indices. By E_H we denote the identity operator in H , i.e. $E_H x = x$, $x \in H$. In obvious cases we may omit the index H . If H_1 is a subspace of H , then $P_{H_1} = P_{H_1}^H$ is an operator of the orthogonal projection

on H_1 in H . By $w.$ - \lim and $u.$ - \lim we denote the limits in the weak and the uniform operator topologies, respectively.

2 A connection between Chumakin's formula and Inin's formula.

The following proposition holds, see [3, p.34].

Proposition 2.1 *Let V be a closed isometric operator in a Hilbert space H . Let $z_0 \in \mathbb{D}$ be fixed. For an arbitrary point $\zeta \in \mathbb{C} \setminus \{0\}$, $\zeta \neq z_0$, the following two conditions are equivalent:*

- (i) $\zeta^{-1} \in \mathcal{M}_r(V)$;
- (ii) $\frac{1-\zeta\overline{z_0}}{\zeta-z_0} \in \mathcal{M}_r(V_{z_0})$.

Proof. (i) \Rightarrow (ii). We may write

$$\begin{aligned} V_{z_0} - \frac{1-\zeta\overline{z_0}}{\zeta-z_0} E_H &= (V - \overline{z_0} E_H)(E_H - z_0 V)^{-1} - \frac{1-\zeta\overline{z_0}}{\zeta-z_0} (E_H - z_0 V)(E_H - z_0 V)^{-1} \\ &= \frac{\zeta(1-|z_0|^2)}{\zeta-z_0} (V - \frac{1}{\zeta} E_H)(E_H - z_0 V)^{-1}. \end{aligned}$$

The operator on the right-hand side has a bounded inverse defined on $(V - \zeta^{-1} E_H)D(V)$.

(ii) \Rightarrow (i). We write:

$$\begin{aligned} V - \frac{1}{\zeta} E_H &= (V_{z_0} + \overline{z_0} E_H)(E_H + z_0 V_{z_0})^{-1} - \frac{1}{\zeta} (E_H + z_0 V_{z_0})(E_H + z_0 V_{z_0})^{-1} \\ &= \frac{\zeta-z_0}{\zeta} \left(V_{z_0} - \frac{1-\zeta\overline{z_0}}{\zeta-z_0} E_H \right) (E_H + z_0 V_{z_0})^{-1}, \end{aligned}$$

and the operator on the right-hand side has a bounded inverse which is defined on $(V_{z_0} - \frac{1-\zeta\overline{z_0}}{\zeta-z_0} E_H)D(V_{z_0})$. \square

Let V be a closed isometric operator in a Hilbert space H , and $z_0 \in \mathbb{D} \setminus \{0\}$ be a fixed point. Consider the following linear fractional transformation:

$$t = t(u) = \frac{u - \overline{z_0}}{1 - z_0 u}, \quad (11)$$

which maps \mathbb{T} on \mathbb{T} , and \mathbb{D} on \mathbb{D} .

Let \widehat{V}_{z_0} be an arbitrary unitary extension of V_{z_0} in a Hilbert space $\widehat{H} \supseteq H$, and \widehat{V} be the corresponding unitary extension of V defined by relation (5).

Choose an arbitrary $u \in \mathbb{T}_e \setminus \{0, \overline{z_0}, \frac{1}{z_0}\}$. Then $t = t(u) \in \mathbb{T}_e \setminus \{0, -\overline{z_0}, -\frac{1}{z_0}\}$. Moreover

$$u \in \mathbb{T}_e \setminus \{0, \overline{z_0}, \frac{1}{z_0}\} \Leftrightarrow t \in \mathbb{T}_e \setminus \{0, -\overline{z_0}, -\frac{1}{z_0}\}. \quad (12)$$

We may write:

$$\begin{aligned} (\widehat{V}_{z_0} - tE_{\widehat{H}})^{-1} &= \left((\widehat{V} - \overline{z_0}E_{\widehat{H}})(E_{\widehat{H}} - z_0\widehat{V})^{-1} \right. \\ &\quad \left. - \frac{u - \overline{z_0}}{1 - z_0u}(E_{\widehat{H}} - z_0\widehat{V})(E_{\widehat{H}} - z_0\widehat{V})^{-1} \right)^{-1} \\ &= \left(\frac{1 - |z_0|^2}{1 - z_0u}(\widehat{V} - uE_{\widehat{H}})(E_{\widehat{H}} - z_0\widehat{V})^{-1} \right)^{-1} \\ &= \frac{1 - z_0u}{1 - |z_0|^2}(E_{\widehat{H}} - z_0\widehat{V})(\widehat{V} - uE_{\widehat{H}})^{-1} \\ &= -\frac{z_0(1 - z_0u)}{1 - |z_0|^2}E_{\widehat{H}} + \frac{(1 - z_0u)^2}{1 - |z_0|^2}(\widehat{V} - uE_{\widehat{H}})^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{1}{t}(E_{\widehat{H}} - \frac{1}{t}\widehat{V}_{z_0})^{-1} &= -\frac{z_0(1 - z_0u)}{1 - |z_0|^2}E_{\widehat{H}} - \frac{(1 - z_0u)^2}{u(1 - |z_0|^2)}(E_{\widehat{H}} - \frac{1}{u}\widehat{V})^{-1}; \\ (E_{\widehat{H}} - \frac{1}{u}\widehat{V})^{-1} &= -\frac{z_0u}{1 - z_0u}E_{\widehat{H}} + \frac{u(1 - |z_0|^2)}{(1 - z_0u)^2t}(E_{\widehat{H}} - \frac{1}{t}\widehat{V}_{z_0})^{-1} \\ &= -\frac{z_0u}{1 - z_0u}E_{\widehat{H}} + \frac{u(1 - |z_0|^2)}{(1 - z_0u)(u - \overline{z_0})}(E_{\widehat{H}} - \frac{1}{t}\widehat{V}_{z_0})^{-1}. \end{aligned}$$

Set $\tilde{u} = \frac{1}{u}$, $\tilde{t} = \frac{1}{t}$. Observe that $\tilde{u} \in \mathbb{T}_e \setminus \{0, \frac{1}{z_0}, z_0\}$, $\tilde{t} \in \mathbb{T}_e \setminus \{0, -z_0, -\frac{1}{z_0}\}$. Moreover

$$\tilde{u} \in \mathbb{T}_e \setminus \{0, \frac{1}{z_0}, z_0\} \Leftrightarrow \tilde{t} \in \mathbb{T}_e \setminus \{0, -z_0, -\frac{1}{z_0}\}, \quad (13)$$

and these conditions are equivalent to conditions from relation (12). Then

$$(E_{\widehat{H}} - \tilde{u}\widehat{V})^{-1} = -\frac{z_0}{\tilde{u} - z_0}E_{\widehat{H}} + \frac{\tilde{u}(1 - |z_0|^2)}{(\tilde{u} - z_0)(1 - \overline{z_0}\tilde{u})}(E_{\widehat{H}} - \tilde{t}\widehat{V}_{z_0})^{-1}.$$

By applying the projection operator $P_{\widehat{H}}$ to the both sides of the last relation, we obtain the following relation:

$$\mathbf{R}_{\tilde{u}}(V) = -\frac{z_0}{\tilde{u} - z_0}E_H + \frac{\tilde{u}(1 - |z_0|^2)}{(\tilde{u} - z_0)(1 - \overline{z_0}\tilde{u})}\mathbf{R}_{\frac{\tilde{u}-z_0}{1-\overline{z_0}\tilde{u}}}(V_{z_0}), \quad \tilde{u} \in \mathbb{T}_e \setminus \{0, \frac{1}{z_0}, z_0\}, \quad (14)$$

where $\mathbf{R}_{\tilde{u}}(V)$, $\mathbf{R}_{\tilde{t}}(V_{z_0})$, are the generalized resolvents of the operators V , V_{z_0} , respectively.

Since $\mathbf{R}_{\tilde{u}}(V)$ is analytic in \mathbb{T}_e , it is uniquely defined by the generalized resolvent $\mathbf{R}_{\tilde{t}}(V_{z_0})$, by relation (14). By the same relation (14), the generalized resolvent $\mathbf{R}_{\tilde{t}}(V_{z_0})$ is uniquely defined by the generalized resolvent $\mathbf{R}_{\tilde{u}}(V)$.

Thus, relation (14) establishes a one-to-one correspondence between all generalized resolvents of V_{z_0} , and all generalized resolvents of V .

Let us apply Chumakin's formula (2) to the operator V_{z_0} :

$$\mathbf{R}_{\tilde{t}}(V_{z_0}) = [E_H - \tilde{t}(V_{z_0} \oplus F(\tilde{t}))]^{-1}, \quad \tilde{t} \in \mathbb{D}, \quad (15)$$

where $F(\tilde{t})$ belongs to the set $\mathcal{S}(N_{z_0}; N_{\frac{1}{\bar{z}_0}})$. Let us restrict relation (14) to $\tilde{u} \in \mathbb{D} \setminus \{0, z_0\}$ what is equivalent to the condition $\tilde{t} \in \mathbb{D} \setminus \{0, -z_0\}$. In this case it also establishes the above-mentioned one-to-one correspondence. By (14),(15) we get

$$\begin{aligned} \mathbf{R}_{\tilde{u}}(V) &= -\frac{z_0}{\tilde{u} - z_0} E_H \\ &+ \frac{\tilde{u}(1 - |z_0|^2)}{(\tilde{u} - z_0)(1 - \bar{z}_0 \tilde{u})} \left[E_H - \frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}} \left(V_{z_0} \oplus F\left(\frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}}\right) \right) \right]^{-1}, \quad \tilde{u} \in \mathbb{D} \setminus \{0, z_0\}, \end{aligned} \quad (16)$$

where $F(\tilde{t}) \in \mathcal{S}(N_{z_0}; N_{\frac{1}{\bar{z}_0}})$. Relation (16) establishes a one-to-one correspondence between all functions $F(\tilde{t})$ from the set $\mathcal{S}(N_{z_0}; N_{\frac{1}{\bar{z}_0}})$, and all generalized resolvents of the operator V .

Set $C(\tilde{u}) = F(\frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}})$, $u \in \mathbb{D}$. Observe that $C(\tilde{u}) \in \mathcal{S}(N_{z_0}; N_{\frac{1}{\bar{z}_0}})$. We may write

$$\begin{aligned} E_H - \frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}} \left(V_{z_0} \oplus F\left(\frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}}\right) \right) &= E_H - \frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}} V_{z_0; C(\tilde{u})}^+ \\ &= (E_H - z_0 V_{C(\tilde{u}); z_0})(E_H - z_0 V_{C(\tilde{u}); z_0})^{-1} \\ &\quad - \frac{\tilde{u} - z_0}{1 - \bar{z}_0 \tilde{u}} (V_{C(\tilde{u}); z_0} - \bar{z}_0 E_H)(E_H - z_0 V_{C(\tilde{u}); z_0})^{-1} \\ &= \frac{1 - |z_0|^2}{1 - \bar{z}_0 \tilde{u}} (E_H - \tilde{u} V_{C(\tilde{u}); z_0})(E_H - z_0 V_{C(\tilde{u}); z_0})^{-1} \end{aligned}$$

By substitution the last relation into relation (16) and after elementary calculations we get:

$$\mathbf{R}_{\tilde{u}}(V) = (E_H - \tilde{u} V_{C(\tilde{u}); z_0})^{-1}, \quad \tilde{u} \in \mathbb{D} \setminus \{0, z_0\}. \quad (17)$$

Of course, for the case $\tilde{u} = 0$ relation (17) is also true. It remains to check relation (17) for the case $\tilde{u} = z_0$, to obtain Inin's formula.

By Chumakin's formula for $\mathbf{R}_{\tilde{u}}(V)$ we see that $(\mathbf{R}_{\tilde{u}}(V))^{-1}$ is an analytic operator-valued function in \mathbb{D} . By (17) we may write

$$(\mathbf{R}_{z_0}(V))^{-1} = u. - \lim_{\tilde{u} \rightarrow z_0} (\mathbf{R}_{\tilde{u}}(V))^{-1} = E_H - u. - \lim_{\tilde{u} \rightarrow z_0} \tilde{u} V_{C(\tilde{u}); z_0}, \quad (18)$$

where the limits are understood in the uniform operator topology.

The operator-valued function $V_{z_0; C(\tilde{u})}^+ = V_{z_0} \oplus C(\tilde{u})$ is analytic in \mathbb{D} , and its values are contractions in H . Then

$$\|(E_H + z_0 V_{z_0; C(\tilde{u})}^+)h\| \geq \|h\| - |z_0| \|V_{z_0; C(\tilde{u})}^+ h\| \geq (1 - |z_0|) \|h\|, \quad h \in H;$$

$$\|(E_H + z_0 V_{z_0; C(\tilde{u})}^+)^{-1}\| \leq \frac{1}{1 - |z_0|}, \quad \tilde{u} \in \mathbb{D}. \quad (19)$$

Using [10, Footnote on page 83] we obtain that the function $(E_H + z_0 V_{z_0; C(\tilde{u})}^+)^{-1}$ is analytic in \mathbb{D} . Therefore $V_{C(\tilde{u}); z_0} = (V_{z_0; C(\tilde{u})}^+ + \overline{z_0} E_H)(E_H + z_0 V_{z_0; C(\tilde{u})}^+)^{-1}$ is analytic in \mathbb{D} , as well. Passing to the limit in relation (18) we get

$$(\mathbf{R}_{z_0}(V))^{-1} = E_H - z_0 V_{C(z_0); z_0}.$$

Therefore relation (17) holds for the case $\tilde{u} = z_0$, and we proved Inin's formula.

3 An analog of McKelvey's results.

The following proposition is an analog of Theorem 3.1 (A),(B) in [4], see also [5, Lemma 1.1].

Proposition 3.1 *Let V be a closed isometric operator in a Hilbert space H , and $\mathbf{F}(\delta)$, $\delta \in \mathfrak{B}(\mathbb{T})$, be its spectral measure. The following two conditions are equivalent:*

- (i) $\mathbf{F}(\Delta) = 0$, for some open arc Δ of \mathbb{T} ;
- (ii) The generalized resolvent $\mathbf{R}_z(V)$, corresponding to the spectral measure $\mathbf{F}(\delta)$, has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \overline{\Delta}$, where $\overline{\Delta} = \{z \in \mathbb{C} : \overline{z} \in \Delta\}$, for some open arc Δ of \mathbb{T} .

Proof. (i) \Rightarrow (ii). In this case relation (1) takes the following form:

$$(\mathbf{R}_z h, g)_H = \int_{\mathbb{T} \setminus \Delta} \frac{1}{1 - z\zeta} d(\mathbf{F}(\cdot)h, g)_H, \quad \forall h, g \in H. \quad (20)$$

Choose an arbitrary $z_0 \in \overline{\Delta}$. Since $\frac{1}{1 - z_0\zeta}$ is bounded and continuous on $\mathbb{T} \setminus \Delta$, there exists an integral

$$I_{z_0}(h, g) := \int_{\mathbb{T} \setminus \Delta} \frac{1}{1 - z_0\zeta} d(\mathbf{F}(\cdot)h, g)_H.$$

Then

$$\begin{aligned} |(\mathbf{R}_z h, h)_H - I_{z_0}(h, h)| &= |z - z_0| \left| \int_{\mathbb{T} \setminus \Delta} \frac{\zeta}{(1 - z\zeta)(1 - z_0\zeta)} d(\mathbf{F}(\cdot)h, h)_H \right| \\ &\leq |z - z_0| \int_{\mathbb{T} \setminus \Delta} \frac{|\zeta|}{|1 - z\zeta||1 - z_0\zeta|} d(\mathbf{F}(\cdot)h, h)_H, \quad z \in \mathbb{T}_e. \end{aligned}$$

There exists a neighborhood $U(z_0)$ of z_0 such that $|z - \bar{\zeta}| \geq M_1 > 0$, $\forall \zeta \in \mathbb{T} \setminus \Delta$, $\forall z \in U(z_0)$. Therefore the integral in the last relation is bounded in $U(z_0)$. Thus, we obtain that

$$(\mathbf{R}_z h, h)_H \rightarrow I_{z_0}(h, h), \quad z \in \mathbb{T}_e, \quad z \rightarrow z_0, \quad \forall h \in H.$$

Using properties of sesquilinear forms we get

$$(\mathbf{R}_z h, g)_H \rightarrow I_{z_0}(h, g), \quad z \in \mathbb{T}_e, \quad z \rightarrow z_0, \quad \forall h, g \in H.$$

Set

$$\mathbf{R}_{\tilde{z}} := w. - \lim_{z \in \mathbb{T}_e, z \rightarrow \tilde{z}} \mathbf{R}_z, \quad \forall \tilde{z} \in \overline{\Delta},$$

where the limit is understood in the weak operator topology. We may write

$$\begin{aligned} \left(\frac{1}{z - z_0} (\mathbf{R}_z - \mathbf{R}_{z_0}) h, h \right)_H &= \int_{\mathbb{T} \setminus \Delta} \frac{\zeta}{(1 - z\zeta)(1 - z_0\zeta)} d(\mathbf{F}(\cdot)h, h)_H, \\ z &\in U(z_0), \quad h \in H. \end{aligned}$$

The function under the integral is bounded in $U(z_0)$, and it tends to $\frac{\zeta}{(1 - z_0\zeta)^2}$. By the Lebesgue convergence theorem we deduce that

$$\lim_{z \rightarrow z_0} \left(\frac{1}{z - z_0} (\mathbf{R}_z - \mathbf{R}_{z_0}) h, h \right)_H = \int_{\mathbb{T} \setminus \Delta} \frac{\zeta}{(1 - z_0\zeta)^2} d(\mathbf{F}(\cdot)h, h)_H;$$

and therefore

$$\lim_{z \rightarrow z_0} \left(\frac{1}{z - z_0} (\mathbf{R}_z - \mathbf{R}_{z_0})h, g \right)_H = \int_{\mathbb{T} \setminus \Delta} \frac{\zeta}{(1 - z_0 \zeta)^2} d(\mathbf{F}(\cdot)h, g)_H,$$

for $h, g \in H$. Consequently, there exists the derivative of \mathbf{R}_z at $z = z_0$.

(ii) \Rightarrow (i). Choose an arbitrary $h \in H$, and consider the function $\sigma_h(t) := (\mathbf{F}_t h, h)_H$, $t \in [0, 2\pi)$, where \mathbf{F}_t is the left-continuous spectral function of V , corresponding to the spectral measure $\mathbf{F}(\delta)$. Also consider the following function:

$$\begin{aligned} f_h(z) &= \frac{1}{2} \int_0^{2\pi} \frac{1 + e^{it}z}{1 - e^{it}z} d\sigma_h(t) = \int_0^{2\pi} \frac{1}{1 - e^{it}z} d\sigma_h(t) - \frac{1}{2} \int_0^{2\pi} d\sigma_h(t) \\ &= \int_0^{2\pi} \frac{1}{1 - e^{it}z} d\sigma_h(t) - \frac{1}{2} \|h\|_H^2 = (\mathbf{R}_z h, h)_H - \frac{1}{2} \|h\|_H^2. \end{aligned} \quad (21)$$

Choose arbitrary numbers t_1, t_2 , $0 \leq t_1 < t_2 \leq 2\pi$, such that

$$l = l(t_1, t_2) = \{z = e^{it} : t_1 \leq t \leq t_2\} \subset \Delta. \quad (22)$$

Suppose additionally that t_1 and t_2 are points of continuity of the function \mathbf{F}_t . By the inversion formula [1, p.50] we may write:

$$\sigma_h(t_2) - \sigma_h(t_1) = \lim_{r \rightarrow 1-0} \int_{t_1}^{t_2} \operatorname{Re} \{f_h(re^{-i\tau})\} d\tau.$$

Observe that

$$\begin{aligned} \operatorname{Re} \{f_h(re^{-i\tau})\} &= \operatorname{Re} \{(\mathbf{R}_{re^{-i\tau}} h, h)_H\} - \frac{1}{2} \|h\|_H^2 \\ &= \frac{1}{2} ((\mathbf{R}_{re^{-i\tau}} + \mathbf{R}_{re^{-i\tau}}^*)h, h)_H - \frac{1}{2} \|h\|_H^2, \quad t_1 \leq \tau \leq t_2. \end{aligned} \quad (23)$$

By (22) we see that $e^{-i\tau}$ belongs to $\overline{\Delta}$, for $t_1 \leq \tau \leq t_2$. Therefore

$$\lim_{r \rightarrow 1-0} ((\mathbf{R}_{re^{-i\tau}} + \mathbf{R}_{re^{-i\tau}}^*)h, h)_H = ((\mathbf{R}_{e^{-i\tau}} + \mathbf{R}_{e^{-i\tau}}^*)h, h)_H. \quad (24)$$

The generalized resolvents have the following property [2]:

$$\mathbf{R}_z^* = E_H - \mathbf{R}_{\frac{1}{\bar{z}}}, \quad z \in \mathbb{T}_e. \quad (25)$$

Passing to the limit in (25) as z tends to $e^{-i\tau}$, we get

$$\mathbf{R}_{e^{-i\tau}}^* = E_H - \mathbf{R}_{e^{-i\tau}}, \quad t_1 \leq \tau \leq t_2. \quad (26)$$

By (23),(24) and (26) we obtain that

$$\lim_{r \rightarrow 1-0} \operatorname{Re} \{f_h(re^{-i\tau})\} = 0, \quad t_1 \leq \tau \leq t_2. \quad (27)$$

Consider the following sector:

$$L(t_1, t_2) = \{z = re^{-it} : t_1 \leq t \leq t_2, 0 \leq r \leq 1\}.$$

The generalized resolvent is analytic at any point of the closed sector $L(t_1, t_2)$. Therefore $\operatorname{Re}(\mathbf{R}_z h, h)$ is continuous and bounded in $L(t_1, t_2)$. By the Lebesgue convergence theorem we conclude that $\sigma_h(t_1) = \sigma_h(t_2)$. If $1 \notin \Delta$ we easily get the required result. In the case $1 \in \Delta$, we write $\Delta = \Delta_1 \cup \{1\} \cup \Delta_2$ where open arcs Δ_1 and Δ_2 do not contain 1. Then $\sigma_h(t)$ is constant in the intervals corresponding to Δ_1 and Δ_2 . Suppose that there exists a non-zero jump of $\sigma_h(t)$ at $t = 0$. By (1) we may write:

$$(\mathbf{R}_z h, h)_H = \int_0^{2\pi} \frac{1}{1 - e^{it}z} d\sigma_h(t) = \int_0^{2\pi} \frac{1}{1 - e^{it}z} d\widehat{\sigma}_h(t) + \frac{1}{1 - z} a, \quad a > 0,$$

where $\widehat{\sigma}_h(t) = \sigma_h(t) + \sigma_h(+0) - \sigma_h(0)$, $t \in [0, 2\pi]$. In a neighborhood of 1 the left-hand side and the first summand of the right-hand side are bounded. We obtained a contradiction. \square

The following theorem is an analog of Theorem 2.1 (A) in [4].

Theorem 3.1 *Let V be a closed isometric operator in a Hilbert space H , and $\mathbf{R}_z(V)$ be an arbitrary generalized resolvent of V . Let $\{\lambda_k\}_{k=1}^\infty$ be a sequence of points of \mathbb{D} , such that $\lambda_k \rightarrow \widehat{\lambda}$, as $k \rightarrow \infty$; $\widehat{\lambda} \in \mathbb{T}$. Suppose that for some $z_0 \in \mathbb{D} \setminus \{0\}$, the function $C(\lambda; z_0)$, corresponding to $\mathbf{R}_z(V)$ by Inin's formula (10), satisfies the following relation:*

$$\exists u. - \lim_{k \rightarrow \infty} C(\lambda_k; z_0) =: C(\widehat{\lambda}; z_0). \quad (28)$$

Then for arbitrary $z'_0 \in \mathbb{D} \setminus \{0\}$, the function $C(\lambda; z'_0)$, corresponding to $\mathbf{R}_z(V)$ by Inin's formula (10), is such that

$$\exists u. - \lim_{k \rightarrow \infty} C(\lambda_k; z'_0) =: C(\widehat{\lambda}; z'_0). \quad (29)$$

In this case $C(\widehat{\lambda}; z'_0)$ is a linear contraction which maps $N_{z'_0}$ into $N_{\frac{1}{z'_0}}$, and the corresponding orthogonal extension $V_{C(\widehat{\lambda}; z'_0); z'_0}$ does not depend on the choice of $z'_0 \in \mathbb{D} \setminus \{0\}$.

Proof. Suppose that relation (28) holds for some $z_0 \in \mathbb{D} \setminus \{0\}$. Choose an arbitrary $z'_0 \in \mathbb{D} \setminus \{0\}$. Comparing Inin's formula for the choices z_0 and z'_0 we conclude that

$$V_{C(\lambda; z_0); z_0} = V_{C(\lambda; z'_0); z'_0}, \quad \lambda \in \mathbb{D}. \quad (30)$$

By (8) we write

$$V_{C(\lambda; z_0); z_0} = \frac{1}{z_0} E_H + \frac{|z_0|^2 - 1}{z_0} \left(E_H + z_0 V_{z_0; C(\lambda; z_0)}^+ \right)^{-1}, \quad \lambda \in \mathbb{D}. \quad (31)$$

By substitution in (30) such expressions for z_0 and z'_0 , and multiplying by $z_0 z'_0$ we get

$$\begin{aligned} & z'_0 E_H + z'_0 (|z_0|^2 - 1) \left(E_H + z_0 V_{z_0; C(\lambda; z_0)}^+ \right)^{-1} \\ &= z_0 E_H + z_0 (|z'_0|^2 - 1) \left(E_H + z'_0 V_{z'_0; C(\lambda; z'_0)}^+ \right)^{-1}. \end{aligned}$$

Then

$$\begin{aligned} & \left(E_H + z'_0 V_{z'_0; C(\lambda; z'_0)}^+ \right)^{-1} = \frac{1}{z_0 (|z'_0|^2 - 1)} \left((z'_0 - z_0) E_H \right. \\ & \quad \left. + z'_0 (|z_0|^2 - 1) \left(E_H + z_0 V_{z_0; C(\lambda; z_0)}^+ \right)^{-1} \right) \\ &= \frac{1 - z'_0 \overline{z_0}}{1 - |z'_0|^2} \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0; C(\lambda; z_0)}^+ \right) \left(E_H + z_0 V_{z_0; C(\lambda; z_0)}^+ \right)^{-1}, \quad \lambda \in \mathbb{D}. \end{aligned} \quad (32)$$

Lemma 3.1 *Let $z_0, z'_0 \in \mathbb{D}$. Then*

$$\left| \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} \right| < 1. \quad (33)$$

Proof. Consider the linear fractional transformation: $w = w(u) = \frac{z_0 - u}{1 - \overline{z_0} u}$. If $|u| = 1$ then $|1 - \overline{z_0} u| = |u(\overline{u} - \overline{z_0})| = |u - z_0|$. Moreover $w(z_0) = 0$. Therefore w maps \mathbb{D} onto \mathbb{D} . \square

By (32),(33) we may write:

$$\begin{aligned} & E_H + z'_0 V_{z'_0; C(\lambda; z'_0)}^+ \\ &= \frac{1 - |z'_0|^2}{1 - z'_0 \overline{z_0}} \left(E_H + z_0 V_{z_0; C(\lambda; z_0)}^+ \right) \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0; C(\lambda; z_0)}^+ \right)^{-1}, \quad \lambda \in \mathbb{D}. \end{aligned} \quad (34)$$

By (6) we may write:

$$V_{z_0;C(\lambda;z_0)}^+ = V_{z_0} \oplus C(\lambda; z_0), \quad \lambda \in \mathbb{D}.$$

By conditions of the theorem it easily follows that $C(\widehat{\lambda}; z_0)$ is a contraction, and

$$\exists u. - \lim_{k \rightarrow \infty} V_{z_0;C(\lambda_k;z_0)}^+ = V_{z_0} \oplus C(\widehat{\lambda}; z_0) = V_{z_0;C(\widehat{\lambda};z_0)}^+. \quad (35)$$

We may write

$$\begin{aligned} & \left\| \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\lambda_k;z_0)}^+ \right)^{-1} - \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\widehat{\lambda};z_0)}^+ \right)^{-1} \right\| \\ & \leq \left\| \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\lambda_k;z_0)}^+ \right)^{-1} \right\| \\ & * \left| \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} \right| \left\| V_{z_0;C(\lambda_k;z_0)}^+ - V_{z_0;C(\widehat{\lambda};z_0)}^+ \right\| \left\| \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\widehat{\lambda};z_0)}^+ \right)^{-1} \right\|. \end{aligned} \quad (36)$$

Since

$$\left| \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} \right| \left\| V_{z_0;C(\lambda_k;z_0)}^+ \right\| \leq \delta < 1,$$

then

$$\begin{aligned} & \left\| \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\lambda_k;z_0)}^+ \right) h \right\| \geq \left\| h \right\| - \left\| \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\lambda_k;z_0)}^+ \right\| \left\| h \right\| \\ & \geq (1 - \delta) \|h\|; \\ & \left\| \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\lambda_k;z_0)}^+ \right)^{-1} \right\| \leq \frac{1}{1 - \delta}. \end{aligned}$$

Passing to the limit in (36) we obtain that

$$u. - \lim_{k \rightarrow \infty} \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\lambda_k;z_0)}^+ \right)^{-1} = \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0;C(\widehat{\lambda};z_0)}^+ \right)^{-1}. \quad (37)$$

By (37),(35),(34) we conclude that there exists

$$u. - \lim_{k \rightarrow \infty} V_{z'_0;C(\lambda_k;z'_0)}^+ = u. - \lim_{k \rightarrow \infty} V_{z'_0} \oplus C(\lambda_k; z'_0) =: V', \quad (38)$$

such that

$$E_H + z'_0 V'$$

$$= \frac{1 - |z'_0|^2}{1 - z'_0 \overline{z_0}} \left(E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right) \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right)^{-1}. \quad (39)$$

By (38) we see that $V'|_{M_{z'_0}} = V_{z'_0}$. Set

$$C(\widehat{\lambda}; z'_0) = V'|_{N_{z'_0}}.$$

Then relation (38) shows that $C(\widehat{\lambda}; z'_0)$ is a linear contraction which maps $N_{z'_0}$ into $N_{\frac{1}{z'_0}}$. Thus, we have

$$V' = V_{z'_0} \oplus C(\widehat{\lambda}; z'_0) = V_{z'_0; C(\widehat{\lambda}; z'_0)}^+. \quad (40)$$

By (38), (40) we easily get that

$$u. - \lim_{k \rightarrow \infty} C(\lambda_k; z'_0) = C(\widehat{\lambda}; z'_0), \quad (41)$$

and (29) is proved.

By (39), (40) we obtain:

$$\begin{aligned} & \left(E_H + z'_0 V_{z'_0; C(\widehat{\lambda}; z'_0)}^+ \right)^{-1} \\ &= \frac{1 - z'_0 \overline{z_0}}{1 - |z'_0|^2} \left(E_H + \frac{z_0 - z'_0}{1 - z'_0 \overline{z_0}} V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right) \left(E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right)^{-1}; \quad (42) \\ & (1 - |z'_0|^2) \left(E_H + z'_0 V_{z'_0; C(\widehat{\lambda}; z'_0)}^+ \right)^{-1} \\ &= \left((1 - z'_0 \overline{z_0}) E_H + (z_0 - z'_0) V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right) \left(E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right)^{-1}; \end{aligned}$$

By subtracting E_H from the both sides of the last relation and by division by $-z'_0$ we get

$$\begin{aligned} & \frac{1}{z'_0} E_H + \frac{|z'_0|^2 - 1}{z'_0} \left(E_H + z'_0 V_{z'_0; C(\widehat{\lambda}; z'_0)}^+ \right)^{-1} \\ &= -\frac{1}{z'_0} \left((1 - z'_0 \overline{z_0}) E_H + (z_0 - z'_0) V_{z_0; C(\widehat{\lambda}; z_0)}^+ - (E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+) \right) \\ & \quad * \left(E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right)^{-1} \\ &= \left(\overline{z_0} E_H + V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right) \left(E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right)^{-1}. \end{aligned}$$

By (7) and (8) we conclude that

$$V_{C(\widehat{\lambda}; z'_0); z'_0} = V_{C(\widehat{\lambda}; z_0); z_0}, \quad \forall z'_0 \in \mathbb{D}. \quad (43)$$

□

The following theorem is an analog of Theorem 2.1 (B) in [4].

Theorem 3.2 *Let V be a closed isometric operator in a Hilbert space H , and $z_0 \in \mathbb{D} \setminus \{0\}$ be a fixed point. Let $\mathbf{R}_z = \mathbf{R}_z(V)$ be an arbitrary generalized resolvent of V , and $C(\lambda; z_0) \in \mathcal{S}(N_{z_0}; N_{\frac{1}{z_0}})$ corresponds to $\mathbf{R}_z(V)$ by Inin's formula (10). $\mathbf{R}_z(V)$ has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$, for some open arc Δ of \mathbb{T} , if and only if the following conditions are satisfied:*

- 1) $C(\lambda; z_0)$ has an extension to the set $\mathbb{D} \cup \Delta$ which is continuous in the uniform operator topology;
- 2) The extended $C(\lambda; z_0)$ maps isometrically N_{z_0} on the whole $N_{\frac{1}{z_0}}$, for all $\lambda \in \Delta$;
- 3) The operator $(E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1}$ exists and it is defined on the whole H , for all $\lambda \in \Delta$.

Proof. *Necessity.* Choose an arbitrary point $\widehat{\lambda} \in \Delta$. Let $z \in \mathbb{D} \setminus \{0\}$ be an arbitrary point, and $C(\lambda; z) \in \mathcal{S}(N_z; N_{\frac{1}{z}})$ corresponds to the generalized resolvent $\mathbf{R}_z(V)$ by Inin's formula (10). Using Inin's formula we may write:

$$\begin{aligned} E_H - zV_{C(\lambda; z); z} &= \frac{z}{\lambda}(E_H - \lambda V_{C(\lambda; z); z}) + \left(1 - \frac{z}{\lambda}\right) E_H \\ &= \frac{z}{\lambda} \mathbf{R}_\lambda^{-1} + \left(1 - \frac{z}{\lambda}\right) E_H = \frac{z}{\lambda} \left[E_H + \left(\frac{\lambda}{z} - 1\right) \mathbf{R}_\lambda \right] \mathbf{R}_\lambda^{-1}, \quad \forall \lambda \in \mathbb{D} \setminus \{0\}. \end{aligned}$$

Therefore

$$\left[E_H + \left(\frac{\lambda}{z} - 1\right) \mathbf{R}_\lambda \right] = \frac{\lambda}{z} (E_H - zV_{C(\lambda; z); z}) \mathbf{R}_\lambda,$$

has a bounded inverse defined on the whole H :

$$\left[E_H + \left(\frac{\lambda}{z} - 1\right) \mathbf{R}_\lambda \right]^{-1} = \frac{z}{\lambda} \mathbf{R}_\lambda^{-1} (E_H - zV_{C(\lambda; z); z})^{-1}, \quad \lambda \in \mathbb{D} \setminus \{0\}. \quad (44)$$

Then

$$(E_H - zV_{C(\lambda; z); z})^{-1} = \frac{\lambda}{z} \mathbf{R}_\lambda \left[E_H + \left(\frac{\lambda}{z} - 1\right) \mathbf{R}_\lambda \right]^{-1}, \quad \lambda \in \mathbb{D} \setminus \{0\}. \quad (45)$$

Choose an arbitrary δ : $0 < \delta < 1$. Assume that $z \in \mathbb{D} \setminus \{0\}$ satisfies the following additional condition:

$$\left| \frac{\hat{\lambda}}{z} - 1 \right| \|\mathbf{R}_{\hat{\lambda}}\| < \delta. \quad (46)$$

Let us check that such points exist. If $\|\mathbf{R}_{\hat{\lambda}}\| = 0$, it is obvious. In the opposite case, we look for $z = \varepsilon \hat{\lambda}$, $0 < \varepsilon < 1$. In this case condition (46) means that

$$\left| \frac{1}{\varepsilon} - 1 \right| < \frac{\delta}{\|\mathbf{R}_{\hat{\lambda}}\|};$$

or $\varepsilon > \frac{1}{1 + \frac{\delta}{\|\mathbf{R}_{\hat{\lambda}}\|}}$.

Then there exists $\left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right]^{-1}$ which is bounded and defined on the whole H . Moreover, by continuity inequality

$$\left| \frac{\lambda}{z} - 1 \right| \|\mathbf{R}_{\lambda}\| < \delta, \quad (47)$$

holds in an open neighborhood $U(\hat{\lambda})$ of $\hat{\lambda}$, and therefore there exists

$$\left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1}, \quad \forall \lambda \in U(\hat{\lambda}), \quad (48)$$

which is bounded and defined on the whole H . We may write

$$\begin{aligned} \left\| \left(E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right) h \right\| &\geq \left| \|h\| - \left| \frac{\lambda}{z} - 1 \right| \|\mathbf{R}_{\lambda} h\| \right| \\ &\geq (1 - \delta) \|h\|, \quad h \in H. \end{aligned}$$

Therefore

$$\left\| \left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1} \right\| \leq \frac{1}{1 - \delta}, \quad \lambda \in U(\hat{\lambda}). \quad (49)$$

Choose an arbitrary sequence $\{\lambda_k\}_{k=1}^{\infty}$ of points in \mathbb{D} , such that $\lambda_k \rightarrow \hat{\lambda}$, as $k \rightarrow \infty$. There exists a number $k_0 \in \mathbb{N}$ such that $\lambda_k \in U(\hat{\lambda}) \cap \mathbb{D}$, $k \geq k_0$. We may write

$$\left\| \left[E_H + \left(\frac{\lambda_k}{z} - 1 \right) \mathbf{R}_{\lambda_k} \right]^{-1} - \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right]^{-1} \right\|$$

$$\leq \left\| \left[E_H + \left(\frac{\lambda_k}{z} - 1 \right) \mathbf{R}_{\lambda_k} \right]^{-1} \right\| \left\| \left(\frac{\lambda_k}{z} - 1 \right) \mathbf{R}_{\lambda_k} - \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right\| \\ * \left\| \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right]^{-1} \right\|.$$

The first factor on the right of the last equality is uniformly bounded by (49). Thus, we get

$$u.-\lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} \left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1} = \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right]^{-1}. \quad (50)$$

By relations (45),(50) we conclude that

$$u.-\lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} (E_H - zV_{C(\lambda;z);z})^{-1} = \frac{\hat{\lambda}}{z} \mathbf{R}_{\hat{\lambda}} \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right]^{-1}. \quad (51)$$

Then there exists the following limit:

$$u.-\lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} V_{z;C(\lambda;z)}^+ = u.-\lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} \left(-\frac{1}{z} E_H + \frac{1-|z|^2}{z} (E_H - zV_{C(\lambda;z);z})^{-1} \right) \\ = -\frac{1}{z} E_H + \frac{1-|z|^2}{z^2} \hat{\lambda} \mathbf{R}_{\hat{\lambda}} \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right]^{-1} =: V'_z, \quad (52)$$

where we used (9). Notice that $V_{z;C(\lambda;z)}^+ = V_z \oplus C(\lambda; z)$. Set $C(\hat{\lambda}; z) = V'_z|_{N_z}$. By (52) we conclude that $C(\hat{\lambda}; z)$ is a linear contraction which maps N_z into $N_{\frac{1}{\bar{z}}}$. Moreover, $V'_z|_{M_z} = V_z$, and therefore

$$V'_z = V_z \oplus C(\hat{\lambda}; z) = V_{z;C(\hat{\lambda};z)}^+.$$

By (52) we easily obtain that

$$u.-\lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} C(\lambda; z) = C(\hat{\lambda}; z). \quad (53)$$

By Theorem 3.1 we conclude that the last relation also holds for z_0 , where $C(\hat{\lambda}; z_0)$ is a linear contraction which maps N_{z_0} into $N_{\frac{1}{\bar{z}_0}}$, and $V_{C(\hat{\lambda};z_0);z_0} = V_{C(\hat{\lambda};z);z}$.

Comparing relation (52) for $V'_z = V_{z;C(\widehat{\lambda};z)}^+$, with relation (9) we get:

$$\frac{\widehat{\lambda}}{z} \mathbf{R}_{\widehat{\lambda}} \left[E_H + \left(\frac{\widehat{\lambda}}{z} - 1 \right) \mathbf{R}_{\widehat{\lambda}} \right]^{-1} = \left(E_H - z V_{C(\widehat{\lambda};z);z} \right)^{-1}, \quad (54)$$

for the above choice of z .

Thus, we have extended by continuity the function $C(\lambda; z_0)$ to the set $\mathbb{D} \cup \Delta$. Let us check that this extension is continuous in the uniform operator topology. It remains to check that for an arbitrary $\widehat{\lambda} \in \Delta$ we have

$$u. - \lim_{\lambda \in \mathbb{D} \cup \Delta, \lambda \rightarrow \widehat{\lambda}} C(\lambda; z_0) = C(\widehat{\lambda}; z_0). \quad (55)$$

We choose $z \in \mathbb{D} \setminus \{0\}$ satisfying (46) and construct a neighborhood $U(\widehat{\lambda})$, as before. For an arbitrary $\lambda \in U(\widehat{\lambda})$ we may write:

$$\begin{aligned} & \left\| \left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1} - \left[E_H + \left(\frac{\widehat{\lambda}}{z} - 1 \right) \mathbf{R}_{\widehat{\lambda}} \right]^{-1} \right\| \\ & \leq \left\| \left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1} \right\| \left\| \left(\frac{\widehat{\lambda}}{z} - 1 \right) \mathbf{R}_{\widehat{\lambda}} - \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right\| \\ & \quad * \left\| \left[E_H + \left(\frac{\widehat{\lambda}}{z} - 1 \right) \mathbf{R}_{\widehat{\lambda}} \right]^{-1} \right\|. \end{aligned}$$

By (49) we obtain that

$$u. - \lim_{\lambda \rightarrow \widehat{\lambda}} \left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1} = \left[E_H + \left(\frac{\widehat{\lambda}}{z} - 1 \right) \mathbf{R}_{\widehat{\lambda}} \right]^{-1}. \quad (56)$$

By (54) we conclude that

$$\frac{\lambda}{z} \mathbf{R}_{\lambda} \left[E_H + \left(\frac{\lambda}{z} - 1 \right) \mathbf{R}_{\lambda} \right]^{-1} = (E_H - z V_{C(\lambda;z);z})^{-1}, \quad \forall \lambda \in (U(\widehat{\lambda}) \cap \overline{\mathbb{D}}) \setminus \{0\}, \quad (57)$$

for the above choice of z . In fact, for an arbitrary $\widetilde{\lambda} \in \Delta \cap U(\widehat{\lambda})$, there exists a neighborhood $\widetilde{U}(\widetilde{\lambda}) \subset U(\widehat{\lambda})$, where inequality (47) holds for the same choice of z . Then repeating the arguments after (47) for $\widetilde{\lambda}$ instead of $\widehat{\lambda}$, we obtain that (57) holds for $\widetilde{\lambda}$. For the points inside \mathbb{D} we may use relation (45).

By relations (56),(57) we get

$$u. - \lim_{\lambda \in \mathbb{D} \cup \Delta, \lambda \rightarrow \hat{\lambda}} (E_H - zV_{C(\lambda; z); z})^{-1} = (E_H - zV_{C(\hat{\lambda}; z); z})^{-1}. \quad (58)$$

Since it was proven that $V_{C(\hat{\lambda}; z); z}$ does not depend on the choice of $z \in \mathbb{D} \setminus \{0\}$ (and for $z \in \mathbb{D}$ this fact follows from Inin's formula), the last relation holds for all $z \in \mathbb{D} \setminus \{0\}$.

By relation (9) we obtain that

$$u. - \lim_{\lambda \in \mathbb{D} \cup \Delta, \lambda \rightarrow \hat{\lambda}} V_{z; C(\lambda; z)}^+ = V_{z; C(\hat{\lambda}; z)}^+, \quad \forall z \in \mathbb{D} \setminus \{0\}, \quad (59)$$

and therefore relation (55) holds. Thus, condition 1) in the statement of the theorem is proven.

By (54) we see that

$$\mathbf{R}_{\hat{\lambda}} = \frac{z}{\hat{\lambda}} (E_H - zV_{C(\hat{\lambda}; z); z})^{-1} \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right], \quad (60)$$

for the above choice of z . Therefore $\mathbf{R}_{\hat{\lambda}}$ has a bounded inverse, defined on the whole H . Then

$$\begin{aligned} E_H &= \frac{z}{\hat{\lambda}} (E_H - zV_{C(\hat{\lambda}; z); z})^{-1} \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right] \mathbf{R}_{\hat{\lambda}}^{-1}; \\ E_H - zV_{C(\hat{\lambda}; z); z} &= \frac{z}{\hat{\lambda}} \left[E_H + \left(\frac{\hat{\lambda}}{z} - 1 \right) \mathbf{R}_{\hat{\lambda}} \right] \mathbf{R}_{\hat{\lambda}}^{-1} \\ &= \frac{z}{\hat{\lambda}} \mathbf{R}_{\hat{\lambda}}^{-1} + \left(1 - \frac{z}{\hat{\lambda}} \right) E_H. \end{aligned}$$

From the last relation we get

$$\mathbf{R}_{\hat{\lambda}} = (E_H - \hat{\lambda}V_{C(\hat{\lambda}; z); z})^{-1}. \quad (61)$$

Since $V_{C(\hat{\lambda}; z); z}$ does not depend on the choice of z , the last relation holds for all $z \in \mathbb{D} \setminus \{0\}$. Consequently, condition 3) in the statement of the theorem is proven.

By the property (25), passing to the limit as $z \rightarrow \hat{\lambda}$, we get

$$\mathbf{R}_{\hat{\lambda}}^* = E_H - \mathbf{R}_{\hat{\lambda}}. \quad (62)$$

On the other hand, by (61) we get

$$\mathbf{R}_{\hat{\lambda}}^* = \left(E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^* \right)^{-1}. \quad (63)$$

By (61)-(63) we see that

$$E_H = \left(E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^* \right)^{-1} + \left(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z} \right)^{-1}, \quad \forall z \in \mathbb{D} \setminus \{0\}. \quad (64)$$

By multiplying the both sides of the last relation by $(E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^*)$ from the left, and by $(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z})$ from the right, we get

$$(E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^*)(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z}) = E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z} + E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^*.$$

After multiplication in the left-hand side and simplification we obtain that

$$V_{C(\hat{\lambda};z);z}^* V_{C(\hat{\lambda};z);z} = E_H, \quad \forall z \in \mathbb{D} \setminus \{0\}.$$

On the other hand, by (64) we may write:

$$\begin{aligned} \left(E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^* \right)^{-1} &= E_H - \left(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z} \right)^{-1} \\ &= -\hat{\lambda} V_{C(\hat{\lambda};z);z} \left(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z} \right)^{-1}; \\ V_{C(\hat{\lambda};z);z} &= -\frac{1}{\bar{\lambda}} \left(E_H - \bar{\lambda} V_{C(\hat{\lambda};z);z}^* \right)^{-1} \left(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z} \right). \end{aligned}$$

Since $\left(E_H - \hat{\lambda} V_{C(\hat{\lambda};z);z} \right)^{-1}$ is defined on the whole H and bounded, we conclude that $R(V_{C(\hat{\lambda};z);z}) = H$. Hence, the operator $V_{C(\hat{\lambda};z);z}$ is unitary in H . Then the corresponding operator $V_{z;C(\hat{\lambda};z)}^+ = V_z \oplus C(\hat{\lambda};z)$ is unitary, as well. In particular, this fact implies that $C(\hat{\lambda};z)$ is isometric and maps N_z on the whole $N_{\frac{1}{\bar{\lambda}}}$. Since z is an arbitrary point from $\mathbb{D} \setminus \{0\}$, we obtain that condition 2) of the theorem is satisfied.

Sufficiency. Let conditions 1)-3) be satisfied. Choose an arbitrary $\hat{\lambda} \in \Delta$. Choose an arbitrary sequence $\{\lambda_k\}_{k=1}^\infty$ of points in $\mathbb{D} \cup \Delta$, such that $\lambda_k \rightarrow \hat{\lambda}$, as $k \rightarrow \infty$. Using (8) we write:

$$E_H - \lambda_k V_{C(\lambda_k;z_0);z_0} = \left(1 - \frac{1}{z_0} \right) E_H - \frac{|z_0|^2 - 1}{z_0} (E_H + z_0 V_{z_0;C(\lambda_k;z_0)}^+)^{-1}$$

$$\begin{aligned}
&= (1 - \lambda_k \overline{z_0}) \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \overline{z_0}} V_{z_0; C(\lambda_k; z_0)}^+ \right] (E_H + z_0 V_{z_0; C(\lambda_k; z_0)}^+)^{-1}; \\
&\quad E_H - \widehat{\lambda} V_{C(\widehat{\lambda}; z_0); z_0} = \\
&= (1 - \widehat{\lambda} \overline{z_0}) \left[E_H + \frac{z_0 - \widehat{\lambda}}{1 - \widehat{\lambda} \overline{z_0}} V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right] (E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+)^{-1}. \quad (65)
\end{aligned}$$

By (65) we write:

$$E_H + \frac{z_0 - \widehat{\lambda}}{1 - \widehat{\lambda} \overline{z_0}} V_{z_0; C(\widehat{\lambda}; z_0)}^+ = \frac{1}{1 - \widehat{\lambda} \overline{z_0}} \left(E_H - \widehat{\lambda} V_{C(\widehat{\lambda}; z_0); z_0} \right) \left(E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right).$$

Since $V_{C(\widehat{\lambda}; z_0); z_0}$ is closed, by condition 3) it follows that there exists $(E_H - \widehat{\lambda} V_{C(\widehat{\lambda}; z_0); z_0})^{-1}$, which is defined on the whole H , and bounded. Therefore there exists $[E_H + \frac{z_0 - \widehat{\lambda}}{1 - \widehat{\lambda} \overline{z_0}} V_{z_0; C(\widehat{\lambda}; z_0)}^+]^{-1}$, which is bounded and defined on the whole H . From (65) it follows that

$$\begin{aligned}
&\left(E_H - \widehat{\lambda} V_{C(\widehat{\lambda}; z_0); z_0} \right)^{-1} \\
&= \frac{1}{1 - \widehat{\lambda} \overline{z_0}} (E_H + z_0 V_{z_0; C(\widehat{\lambda}; z_0)}^+) \left[E_H + \frac{z_0 - \widehat{\lambda}}{1 - \widehat{\lambda} \overline{z_0}} V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right]^{-1}. \quad (66)
\end{aligned}$$

For points λ_k which belong to Δ we may apply the same argument, while for points λ_k from \mathbb{D} we can use Lemma 3.1, to obtain an analogous representation:

$$\begin{aligned}
&\left(E_H - \lambda_k V_{C(\lambda_k; z_0); z_0} \right)^{-1} \\
&= \frac{1}{1 - \lambda_k \overline{z_0}} (E_H + z_0 V_{z_0; C(\lambda_k; z_0)}^+) \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \overline{z_0}} V_{z_0; C(\lambda_k; z_0)}^+ \right]^{-1}, \quad k \in \mathbb{N}. \quad (67)
\end{aligned}$$

Observe that by condition 1) we have:

$$\begin{aligned}
&\left\| V_{z_0; C(\lambda_k; z_0)}^+ - V_{z_0; C(\widehat{\lambda}; z_0)}^+ \right\| = \sup_{h \in H, \|h\|=1} \left\| (C(\lambda_k; z_0) - C(\widehat{\lambda}; z_0)) P_{N_{z_0}} h \right\| \\
&\leq \left\| C(\lambda_k; z_0) - C(\widehat{\lambda}; z_0) \right\| \rightarrow 0, \quad k \rightarrow \infty; \\
&u. - \lim_{k \rightarrow \infty} V_{z_0; C(\lambda_k; z_0)}^+ = V_{z_0; C(\widehat{\lambda}; z_0)}^+. \quad (68)
\end{aligned}$$

Let us check that there exists an open neighborhood $U_1(\hat{\lambda})$ of $\hat{\lambda}$, and a number $K > 0$ such that

$$\left\| \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \bar{z}_0} V_{z_0; C(\lambda_k; z_0)}^+ \right]^{-1} \right\| \leq K, \quad \forall \lambda_k : \lambda_k \in U_1(\hat{\lambda}). \quad (69)$$

The latter condition is equivalent to the following condition:

$$\left\| \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \bar{z}_0} V_{z_0; C(\lambda_k; z_0)}^+ \right] g \right\| \geq \frac{1}{K} \|g\|, \quad \forall g \in H, \quad \forall \lambda_k : \lambda_k \in U_1(\hat{\lambda}). \quad (70)$$

Suppose to the contrary that condition (70) is not true. Choose a sequence of open balls $U^n(\hat{\lambda})$ with the centrum at $\hat{\lambda}$ and radius $\frac{1}{n}$; and set $K_n = n$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exist elements $g_n \in H$, and $\lambda_{k_n} \in U^n(\hat{\lambda})$, $k_n \in \mathbb{N}$, such that:

$$\left\| \left[E_H + \frac{z_0 - \lambda_{k_n}}{1 - \lambda_{k_n} \bar{z}_0} V_{z_0; C(\lambda_{k_n}; z_0)}^+ \right] g_n \right\| < \frac{1}{n} \|g_n\|. \quad (71)$$

It is clear that g_n are all non-zero. Set $\hat{g}_n = \frac{g_n}{\|g_n\|_H}$, $n \in \mathbb{N}$. Then

$$\left\| \left[E_H + \frac{z_0 - \lambda_{k_n}}{1 - \lambda_{k_n} \bar{z}_0} V_{z_0; C(\lambda_{k_n}; z_0)}^+ \right] \hat{g}_n \right\| < \frac{1}{n}. \quad (72)$$

Since $|\lambda_{k_n} - \hat{\lambda}| < \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \lambda_{k_n} = \hat{\lambda}$. Then we may write

$$\begin{aligned} \frac{1}{n} &> \left\| \left[E_H + \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right] \hat{g}_n \right. \\ &+ \left. \left(\frac{z_0 - \lambda_{k_n}}{1 - \lambda_{k_n} \bar{z}_0} V_{z_0; C(\lambda_{k_n}; z_0)}^+ - \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right) \hat{g}_n \right\| \\ &\geq \left\| \left[E_H + \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right] \hat{g}_n \right\| \\ &\quad - \left\| \frac{z_0 - \lambda_{k_n}}{1 - \lambda_{k_n} \bar{z}_0} V_{z_0; C(\lambda_{k_n}; z_0)}^+ - \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right\| \\ &\geq L - \left\| \frac{z_0 - \lambda_{k_n}}{1 - \lambda_{k_n} \bar{z}_0} V_{z_0; C(\lambda_{k_n}; z_0)}^+ - \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right\|, \quad L > 0, \end{aligned} \quad (73)$$

for sufficiently large n , since $[E_H + \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda}\bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+]$ has a bounded inverse on the whole H , and the norm in the right-hand side tends to zero. Passing to the limit in relation (73) as $n \rightarrow \infty$, we obtain a contradiction. Consequently, there exists an open neighborhood $U_1(\hat{\lambda})$ of $\hat{\lambda}$, and a number $K > 0$ such that inequality (69) holds. We may write:

$$\begin{aligned} & \left\| \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \bar{z}_0} V_{z_0; C(\lambda_k; z_0)}^+ \right]^{-1} - \left[E_H + \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right]^{-1} \right\| \\ & \leq \left\| \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \bar{z}_0} V_{z_0; C(\lambda_k; z_0)}^+ \right]^{-1} \right\| \left\| \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ - \frac{z_0 - \lambda_k}{1 - \lambda_k \bar{z}_0} V_{z_0; C(\lambda_k; z_0)}^+ \right\| \\ & \quad * \left\| \left[E_H + \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right]^{-1} \right\|. \end{aligned}$$

By (69) we conclude that

$$u. - \lim_{k \rightarrow \infty} \left[E_H + \frac{z_0 - \lambda_k}{1 - \lambda_k \bar{z}_0} V_{z_0; C(\lambda_k; z_0)}^+ \right]^{-1} = \left[E_H + \frac{z_0 - \hat{\lambda}}{1 - \hat{\lambda} \bar{z}_0} V_{z_0; C(\hat{\lambda}; z_0)}^+ \right]^{-1}. \quad (74)$$

By (66),(67),(68),(74) we conclude that

$$u. - \lim_{k \rightarrow \infty} (E_H - \lambda_k V_{C(\lambda_k; z_0); z_0})^{-1} = (E_H - \hat{\lambda} V_{C(\hat{\lambda}; z_0); z_0})^{-1}, \quad (75)$$

and therefore

$$u. - \lim_{\lambda \in \mathbb{D} \cup \Delta, \lambda \rightarrow \hat{\lambda}} (E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} = (E_H - \hat{\lambda} V_{C(\hat{\lambda}; z_0); z_0})^{-1}. \quad (76)$$

By Inin's formula, for $\lambda \in \mathbb{D}$, we have $(E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} = \mathbf{R}_\lambda$. Thus, relation (76) shows that the operator-valued function \mathbf{R}_λ , $\lambda \in \mathbb{D}$, has a continuation to the set $\mathbb{D} \cup \Delta$, which is continuous in the uniform operator topology.

Choose an arbitrary $h \in H$ and consider an analytic function

$$f(\lambda) = f_h(\lambda) = (\mathbf{R}_\lambda h, h), \quad \lambda \in \mathbb{D}. \quad (77)$$

Then $f(\lambda)$ has a continuous extension to $\mathbb{D} \cup \Delta$, which is equal to

$$f(\lambda) = \left((E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} h, h \right), \quad \lambda \in \mathbb{D} \cup \Delta.$$

Let us check that

$$(E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} = E_H - (E_H - \bar{\lambda} V_{C(\lambda; z_0); z_0}^*)^{-1}, \quad \forall \lambda \in \Delta. \quad (78)$$

Choose an arbitrary $\lambda \in \Delta$. By condition 2) of the theorem we conclude that $V_{C(\lambda; z_0); z_0}$ is unitary. Then

$$(E_H - \bar{\lambda} V_{C(\lambda; z); z}^*)(E_H - \lambda V_{C(\lambda; z); z}) = E_H - \lambda V_{C(\lambda; z); z} + E_H - \bar{\lambda} V_{C(\lambda; z); z}^*. \quad (79)$$

To verify the last relation, it is sufficient to make multiplication in the left-hand side and a simplification. Multiplying (79) by $(E_H - \bar{\lambda} V_{C(\lambda; z_0); z_0}^*)^{-1}$ from the left, and by $(E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1}$ from the right, we easily get (78).

We may write:

$$\begin{aligned} \overline{f(\lambda)} &= \left(h, (E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} h \right) = \left((E_H - \bar{\lambda} V_{C(\lambda; z_0); z_0}^*)^{-1} h, h \right) \\ &= (h, h) - \left((E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} h, h \right) = (h, h) - f(\lambda); \\ \operatorname{Re} f(\lambda) &= \frac{1}{2}(h, h), \quad \lambda \in \Delta. \end{aligned} \quad (80)$$

Set $g(\lambda) = if(\lambda) - \frac{i}{2}(h, h)$, $\lambda \in \mathbb{D} \cup \Delta$. Then $\operatorname{Im} g(\lambda) = 0$. Consequently, by the Schwarz principle, $g(\lambda)$ admits an analytic continuation $\tilde{g}(\lambda) = \tilde{g}_h(\lambda)$ to the set $\mathbb{D} \cup \Delta \cup \mathbb{D}_e$. Moreover, we have

$$\tilde{g}(\lambda) = \overline{\tilde{g}\left(\frac{1}{\bar{\lambda}}\right)}, \quad \lambda \in \mathbb{D}_e. \quad (81)$$

Then

$$\tilde{f}(\lambda) = \tilde{f}_h(\lambda) := \frac{1}{i}\tilde{g}(\lambda) + \frac{1}{2}(h, h), \quad \lambda \in \mathbb{D} \cup \Delta \cup \mathbb{D}_e,$$

is an analytic continuation of $f(\lambda)$. By (81) we get

$$\tilde{f}(\lambda) = -\overline{\tilde{f}\left(\frac{1}{\bar{\lambda}}\right)} + (h, h), \quad \lambda \in \mathbb{D}_e. \quad (82)$$

Using (25) we may write:

$$\begin{aligned} \tilde{f}(\lambda) &= -\overline{\left(\mathbf{R}_{\frac{1}{\bar{\lambda}}} h, h\right)_H} + (h, h)_H = -(h, (E_H - \mathbf{R}_\lambda^*)h)_H + (h, h)_H \\ &= (\mathbf{R}_\lambda h, h)_H, \quad \lambda \in \mathbb{D}_e. \end{aligned} \quad (83)$$

Set

$$R_\lambda(h, g) = \frac{1}{4} \left(\tilde{f}_{h+g}(\lambda) - \tilde{f}_{h-g}(\lambda) + i\tilde{f}_{h+ig}(\lambda) - i\tilde{f}_{h-ig}(\lambda) \right),$$

$$h, g \in H, \quad \lambda \in \mathbb{D} \cup \Delta \cup \mathbb{D}_e. \quad (84)$$

Observe that $R_\lambda(h, g)$ is an analytic function of λ on $\mathbb{D} \cup \Delta \cup \mathbb{D}_e$. By (77), (83) we see that

$$R_\lambda(h, g) = (\mathbf{R}_\lambda h, g)_H, \quad h, g \in H, \quad \lambda \in \mathbb{D} \cup \mathbb{D}_e. \quad (85)$$

By (76) we see that

$$\begin{aligned} R_{\hat{\lambda}}(h, g) &= \lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} (\mathbf{R}_\lambda h, g)_H \\ &= \lim_{\lambda \in \mathbb{D}, \lambda \rightarrow \hat{\lambda}} \left((E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} h, g \right)_H \\ &= \left((E_H - \hat{\lambda} V_{C(\hat{\lambda}; z_0); z_0})^{-1} h, g \right)_H, \quad h, g \in H, \quad \hat{\lambda} \in \Delta. \end{aligned} \quad (86)$$

Then the following operator-valued function:

$$T_\lambda = \begin{cases} \mathbf{R}_\lambda, & \lambda \in \mathbb{D} \cup \mathbb{D}_e \\ (E_H - \lambda V_{C(\lambda; z_0); z_0})^{-1} & \lambda \in \Delta \end{cases},$$

is an extension of \mathbf{R}_λ , which is analytic in the weak operator topology, and therefore in the uniform operator topology. \square

Corollary 3.1 *Theorem 3.2 remains valid for the choice $z_0 = 0$.*

Proof. Let V be a closed isometric operator in a Hilbert space H . Let $\mathbf{R}_z(V)$ be an arbitrary generalized resolvent of V , and $F(\lambda) = C(\lambda; 0) \in \mathcal{S}(N_0; N_\infty)$ corresponds to $\mathbf{R}_z(V)$ by Inin's formula (10) for $z_0 = 0$, which in this case coincides with Chumakin's formula (2). Consider an arbitrary open arc Δ of \mathbb{T} .

Choose an arbitrary point $z_0 \in \mathbb{D} \setminus \{0\}$. Consider the following isometric operator

$$\mathbf{V} = (V + \overline{z_0} E_H)(E_H + z_0 V)^{-1}, \quad D(\mathbf{V}) = (E_H + z_0 V)D(V).$$

Then

$$V = (\mathbf{V} - \overline{z_0} E_H)(E_H - z_0 \mathbf{V})^{-1} = \mathbf{V}_{z_0}.$$

Recall that generalized resolvents of \mathbf{V} and \mathbf{V}_{z_0} are related by (14) and this correspondence is bijective. Let $\mathbf{R}_z(\mathbf{V})$ be the generalized resolvent which corresponds by (14) to the generalized resolvent $\mathbf{R}_z(\mathbf{V}_{z_0}) = \mathbf{R}_z(V)$.

By (14) we see that $\mathbf{R}_{\tilde{t}}(\mathbf{V}_{z_0})$ has a limit as $\tilde{t} \rightarrow \tilde{t}_0 \in \Delta$, if and only if $\mathbf{R}_{\tilde{u}}(\mathbf{V})$ has a limit as $\tilde{u} \rightarrow \tilde{u}_0 \in \Delta_1$, where

$$\Delta_1 = \left\{ \tilde{u} : \tilde{u} = \frac{\tilde{t} + z_0}{1 + \overline{z_0}\tilde{t}}, \tilde{t} \in \Delta \right\}.$$

Thus, $\mathbf{R}_{\tilde{t}}(\mathbf{V}_{z_0})$ has an extension by continuity to $\mathbb{T}_e \cup \Delta$, iff $\mathbf{R}_{\tilde{u}}(\mathbf{V})$ has an extension by continuity to $\mathbb{T}_e \cup \Delta_1$. The extended values are related by (14), as well. From (14) we see that the extension of $\mathbf{R}_{\tilde{t}}(\mathbf{V}_{z_0})$ is analytic iff the extension of $\mathbf{R}_{\tilde{u}}(\mathbf{V})$ is analytic. Consequently, $\mathbf{R}_{\tilde{t}}(V) = \mathbf{R}_{\tilde{t}}(\mathbf{V}_{z_0})$ has an analytic extension to $\mathbb{T}_e \cup \Delta$, if and only if $\mathbf{R}_{\tilde{u}}(\mathbf{V})$ has an analytic extension to $\mathbb{T}_e \cup \Delta_1$.

By Theorem 3.2, $\mathbf{R}_{\tilde{u}}(\mathbf{V})$ has an analytic extension to $\mathbb{T}_e \cup \Delta_1$ iff

- 1) $C(\lambda; z_0)$ has an extension to the set $\mathbb{D} \cup \Delta_1$ which is continuous in the uniform operator topology;
- 2) The extended $C(\lambda; z_0)$ maps isometrically N_{z_0} on the whole $N_{\frac{1}{\overline{z_0}}}$, for all $\lambda \in \Delta_1$;
- 3) The operator $(E_H - \lambda \mathbf{V}_{C(\lambda; z_0); z_0})^{-1}$ exists and it is defined on the whole H , for all $\lambda \in \Delta_1$,

where $C(\lambda; z_0) \in \mathcal{S}(N_{z_0}; N_{\frac{1}{\overline{z_0}}})$ corresponds to $\mathbf{R}_z(\mathbf{V})$ by Inin's formula (10).

Recall that $C(\lambda; z_0)$ is related to $F(\tilde{t})$ in the following way:

$$C(\tilde{u}) = F\left(\frac{\tilde{u} - z_0}{1 - \overline{z_0}\tilde{u}}\right), \quad u \in \mathbb{T}_e.$$

By using this relation we easily get that condition 1) is equivalent to

1') $F(\tilde{t})$ has an extension to the set $\mathbb{D} \cup \Delta$ which is continuous in the uniform operator topology;

and condition 2) is equivalent to

2') The extended $F(\tilde{t})$ maps isometrically N_0 on the whole N_∞ , for all $\tilde{t} \in \Delta$.

By (65) we conclude that $(E_H - \lambda \mathbf{V}_{C(\lambda; z_0); z_0})^{-1}$ exists and is defined on the whole H , for all $\lambda \in \Delta_1$, iff

$$\left[E_H - \frac{\lambda - z_0}{1 - \overline{\lambda} z_0} (\mathbf{V}_{z_0} \oplus C(\lambda; z_0)) \right]^{-1} = [E_H - \tilde{t} (\mathbf{V}_{z_0} \oplus F(\tilde{t}))]^{-1},$$

exists and is defined on the whole H , for all $\tilde{t} \in \Delta$. \square

4 Some decompositions of a Hilbert space.

The following result appeared in [7, Lemma 1]. However, its proof was based on Shmulyan's lemma. It seems that no correct proof of this lemma ever appeared published. An attempt to prove Shmulyan's lemma was performed by L.A. Shtraus in [6, Lemma]. However, the proof was not complete. We shall use the idea of L.A. Shtraus to prove the following weaker result.

Theorem 4.1 *Let V be a closed isometric operator in a Hilbert space H . Let $\zeta \in \mathbb{T}$, and ζ^{-1} be a point of the regular type of V . Then the following decompositions are valid:*

$$D(V) \dot{+} N_\zeta = H; \quad (87)$$

$$R(V) \dot{+} N_\zeta = H. \quad (88)$$

Proof. At first, we suppose that $\zeta = 1$. Let us check that

$$\|Vf + g\|_H = \|f + g\|_H, \quad f \in D(V), \quad g \in N_1. \quad (89)$$

In fact, we may write:

$$\begin{aligned} \|Vf + g\|_H^2 &= (Vf, Vf)_H + (Vf, g)_H + (g, Vf)_H + (g, g)_H \\ &= \|f\|_H^2 + (Vf, g)_H + (g, Vf)_H + \|g\|_H^2. \end{aligned}$$

Since $0 = ((E - V)f, g)_H = (f, g)_H - (Vf, g)_H$, we get

$$(Vf, g) = (f, g), \quad f \in D(V), \quad g \in N_1, \quad (90)$$

and therefore

$$\|Vf + g\|_H^2 = \|f\|_H^2 + (f, g)_H + (g, f)_H + \|g\|_H^2 = \|f + g\|_H^2.$$

Consider the following operator:

$$U(f + g) = Vf + g, \quad f \in D(V), \quad g \in N_1. \quad (91)$$

Let us check that this operator is well-defined, with the domain $D(U) = D(V) + N_1$. Let $h \in D(U)$ has two representations:

$$h = f_1 + g_1 = f_2 + g_2, \quad f_1, f_2 \in D(V), \quad g_1, g_2 \in N_1.$$

Then using (89) we may write

$$\|Vf_1 + g_1 - (Vf_2 + g_2)\|_H^2 = \|V(f_1 - f_2) + (g_1 - g_2)\|_H^2 = \|f_1 - f_2 + g_1 - g_2\|_H^2 = 0.$$

Thus, U is well-defined. Moreover, U is linear and using (90) we write:

$$\begin{aligned} (U(f+g), U(\tilde{f}+\tilde{g})) &= (Vf+g, V\tilde{f}+\tilde{g}) = (Vf, V\tilde{f}) + (Vf, \tilde{g}) + (g, V\tilde{f}) + (g, \tilde{g}) \\ &= (f, \tilde{f}) + (f, \tilde{g}) + (g, \tilde{f}) + (g, \tilde{g}) = (f+g, \tilde{f}+\tilde{g}), \end{aligned}$$

for $f, \tilde{f} \in D(V)$, $g, \tilde{g} \in N_1$. Therefore U is isometric.

Suppose that there exists $h \in H$, $h \neq 0$, $h \in D(V) \cap N_1$. Then

$$0 = U0 = U(h + (-h)) = Vh - h = (V - E_H)h,$$

and this contradicts to the fact that 1 is a point of the regular type of V . Therefore

$$D(V) \cap N_1 = \{0\}. \quad (92)$$

Notice that we do not know, a priori, that $D(U)$ is closed. Consider the following operator W :

$$W = U|_S,$$

where

$$S = \{h \in D(U) : h \perp N_1\} = D(U) \cap M_1.$$

Thus, W is an isometric operator with the domain $D(W) = D(U) \cap M_1$. Choose an arbitrary element $g \in D(U)$. Then $g = g_{M_1} + g_{N_1}$, $g_{M_1} \in M_1$, $g_{N_1} \in N_1 \subseteq D(U)$. Therefore $g_{M_1} = P_{M_1}^H g \in D(U)$, $g_{M_1} \perp N_1$. Thus, $g_{M_1} \in D(W)$;

$$P_{M_1}^H D(U) \subseteq D(W). \quad (93)$$

On the other hand, choose an arbitrary $h \in D(W)$. Then $h \in D(U) \cap M_1$, and therefore $h = P_{M_1}^H h \in P_{M_1}^H D(U)$. Consequently, we have

$$D(W) = P_{M_1}^H D(U) = P_{M_1}^H (D(V) + N_1) = P_{M_1}^H D(V) \subseteq M_1. \quad (94)$$

Choose an arbitrary $f \in D(V)$. Let $f = f_{M_1} + f_{N_1}$, $f_{M_1} \in M_1$, $f_{N_1} \in N_1$. Then $f - f_{N_1} \in D(U)$, and $f - f_{N_1} \perp N_1$, i.e. $f - f_{N_1} \in D(W)$. We may write

$$(W - E_H)(f - f_{N_1}) = (U - E_H)(f - f_{N_1}) = U(f - f_{N_1}) - f + f_{N_1} = Vf - f;$$

$$(W - E_H)D(W) \supseteq (V - E_H)D(V) = M_1. \quad (95)$$

On the other hand, choose an arbitrary $w \in D(W)$, $w = w_{D(V)} + w_{N_1}$, $w_{D(V)} \in D(V)$, $w_{N_1} \in N_1$. Since $w \perp N_1$, we may write:

$$w = P_{M_1}^H w = P_{M_1}^H w_{D(V)}. \quad (96)$$

We may write

$$(W - E_H)w = Uw - w = Vw_{D(V)} + w_{N_1} - w_{D(V)} - w_{N_1} = (V - E_H)w_{D(V)};$$

$$(W - E_H)D(W) \subseteq (V - E_H)D(V) = M_1.$$

From the last relation and (95) we obtain:

$$(W - E_H)D(W) = (V - E_H)D(V) = M_1. \quad (97)$$

Moreover, if $(W - E_H)w = 0$, then $(V - E_H)w_{D(V)} = 0$; and therefore $w_{D(V)} = 0$. By (96) this implies $w = 0$. Consequently, there exists $(W - E_H)^{-1}$. By (97) we get

$$D(W) = (W - E_H)^{-1}M_1. \quad (98)$$

Since $D(W) \subseteq M_1$, using (97), we may write:

$$Ww = (W - E_H)w + w \in M_1;$$

$$D(W) \subseteq M_1, \quad WD(W) \subseteq M_1. \quad (99)$$

Consider the closure \overline{W} of W with the domain $\overline{D(W)}$. By (99) we see that

$$D(\overline{W}) \subseteq M_1, \quad \overline{W}D(\overline{W}) \subseteq M_1. \quad (100)$$

Then

$$(\overline{W} - E_H)D(\overline{W}) \subseteq M_1.$$

On the other hand, by (97) we have

$$(\overline{W} - E_H)D(\overline{W}) \supseteq (\overline{W} - E_H)D(W) = (W - E_H)D(W) = M_1.$$

Thus, we conclude that

$$(\overline{W} - E_H)D(\overline{W}) = M_1. \quad (101)$$

Let us check that there exists $(\overline{W} - E_H)^{-1}$. Suppose to the contrary that there exists $h \in D(\overline{W})$, $h \neq 0$, such that $(\overline{W} - E_H)h = 0$. By the definition of the closure, there exists a sequence $h_n \in D(W)$, $n \in \mathbb{N}$, which converges

to h , and $Wh_n \rightarrow \overline{W}h$, as $n \rightarrow \infty$. Then $(W - E_H)h_n \rightarrow \overline{W}h - h = 0$, as $n \rightarrow \infty$. Let $h_n = h_{1;n} + h_{2;n}$, $h_{1;n} \in D(V)$, $h_{2;n} \in N_1$, $n \in \mathbb{N}$. Then $(W - E_H)h_n = Uh_n - h_n = Vh_{1;n} + h_{2;n} - h_{1;n} - h_{2;n} = (V - E_H)h_{1;n}$. Therefore $(V - E_H)h_{1;n} \rightarrow 0$, as $n \rightarrow \infty$. Since $(V - E_H)$ has a bounded inverse, we conclude that $h_{1;n} \rightarrow 0$, as $n \rightarrow \infty$. Then $h_n = P_{M_1}^H h_n = P_{M_1}^H h_{1;n} \rightarrow 0$, as $n \rightarrow \infty$. Therefore, we get $h = 0$, and this contradicts to our assumption.

Thus, there exists $(\overline{W} - E_H)^{-1} \supseteq (W - E_H)^{-1}$. By (101),(97), we conclude that

$$(\overline{W} - E_H)^{-1} = (W - E_H)^{-1},$$

and therefore

$$\overline{W} = W.$$

Thus W may be viewed as a closed isometric operator in a Hilbert space M_1 . Then $(W - E_H)^{-1}$ is closed, and it is defined on M_1 . Therefore $(W - E_H)^{-1}$ is bounded. This means that 1 is a regular point of W . Therefore W is a unitary operator in M_1 . In particular, this implies that $D(W) = R(W) = M_1$.

By the definition of W we conclude that $D(W) = M_1 \subseteq D(U)$, and $U|_{M_1} = W$. On the other hand, $U|_{N_1} = E_{N_1}$. Then $D(U) = H$, and

$$U = W \oplus E_{N_1}.$$

Thus, U is a unitary operator. In particular, $D(U) = R(U) = H$, which means that

$$D(V) + N_1 = H, \quad R(V) + N_1 = H. \quad (102)$$

The first sum is direct by (92). Suppose that $h \in R(V) \cap N_1$. Then $h = Vf$, $f \in D(V)$, and we may write:

$$0 = Vf + (-h) = U(f + (-h)).$$

Since U is unitary, we get $f = h = Vf$, $(V - E_H)f = 0$, and therefore $f = 0$, and $h = 0$. Thus, the second sum in (102) is direct, as well. So, we proved the theorem for the case $\zeta = 1$.

In the general case, we can apply the proven part of the theorem to $\widehat{V} := \zeta V$. \square

Corollary 4.1 *In conditions of Theorem 4.1 the following decompositions are valid:*

$$\overline{(H \ominus D(V)) \dot{+} M_\zeta} = H; \quad (103)$$

$$\overline{(H \ominus R(V)) \dot{+} M_\zeta} = H. \quad (104)$$

Proof. The proof is based on the following lemma.

Lemma 4.1 *Let M_1 and M_2 be two subspaces in a Hilbert space H , such that $M_1 \cap M_2 = \{0\}$, and*

$$M_1 \dot{+} M_2 = H. \quad (105)$$

Then

$$\overline{(H \ominus M_1) \dot{+} (H \ominus M_2)} = H. \quad (106)$$

Proof. Suppose that $h \in H$, is such that $h \in ((H \ominus M_1) \cap (H \ominus M_2))$. Then $h \perp M_1$, $h \perp M_2$, and therefore $h \perp (M_1 + M_2)$, $h \perp H$, $h = 0$.

Suppose that $g \in H$, is such that $g \perp ((H \ominus M_1) \dot{+} (H \ominus M_2))$. Then $g \in M_1$, $g \in M_2$, and therefore $g = 0$. \square

By applying the lemma with $M_1 = D(V)$, $M_2 = N_\zeta$, and $M_1 = R(V)$, $M_2 = N_\zeta$, we complete the proof of the corollary. \square

5 Isometric operators with gaps.

Now we shall present full proofs, some slight correction and generalization of Ryabtseva results [7].

For the sake of convenience we put here the proofs of the following lemmas, see Lemma 2 and its corollary, and Lemma 3 in [7].

Lemma 5.1 *Let V be a closed isometric operator in a Hilbert space H , and $\zeta \in \mathbb{T}$. Then the following equality holds:*

$$VP_{M_0}^H f = \zeta^{-1} P_{M_\infty}^H f, \quad \forall f \in N_\zeta, \quad (107)$$

and therefore

$$\|P_{M_0}^H f\| = \|P_{M_\infty}^H f\|, \quad \forall f \in N_\zeta; \quad (108)$$

$$\|P_{N_0}^H f\| = \|P_{N_\infty}^H f\|, \quad \forall f \in N_\zeta. \quad (109)$$

Proof. Choose an arbitrary $f \in N_\zeta$. For an arbitrary $u \in D(V) = M_0$ we may write:

$$\begin{aligned} \left(\zeta^{-1} f - VP_{D(V)}^H f, Vu \right)_H &= \zeta^{-1} (f, Vu)_H - \left(P_{D(V)}^H f, u \right)_H \\ &= (f, \zeta Vu)_H - (f, u)_H = (f, (\zeta V - E_H)u)_H = 0. \end{aligned}$$

Therefore $(\zeta^{-1} f - VP_{M_0}^H f) \perp M_\infty$. By applying $P_{M_\infty}^H$ to this element we get (107). Relation (108) is obvious, since V is isometric, and (109) easily follows. \square

Lemma 5.2 *Let V be a closed isometric operator in a Hilbert space H , and C be a linear bounded operator in H , $D(C) = N_0$, $R(C) \subseteq N_\infty$. Let $\zeta \in \mathbb{T}$, and ζ^{-1} be an eigenvalue of $V_{0;C}^+ = V \oplus C$. If $f \in H$, $f \neq 0$, is an eigenvector of $V_{0;C}^+$ corresponding to ζ^{-1} , then $f \in N_\zeta$, and*

$$CP_{N_0}^H f = \zeta^{-1} P_{N_\infty}^H f. \quad (110)$$

Proof. Let f be an eigenvector of $V_{0;C}^+$ corresponding to $\zeta^{-1} \in \mathbb{T}$:

$$(V \oplus C)f = VP_{M_0}^H f + CP_{N_0}^H f = \zeta^{-1} (P_{M_\infty}^H f + P_{N_\infty}^H f).$$

By the orthogonality of the summands we see that the last relation is equivalent to the following relations

$$VP_{M_0}^H f = \zeta^{-1} P_{M_\infty}^H f; \quad (111)$$

$$CP_{N_0}^H f = \zeta^{-1} P_{N_\infty}^H f, \quad (112)$$

and therefore (110) follows. Relation (111) implies that $P_{M_\infty}^H (\zeta^{-1} f - VP_{M_0}^H f) = 0$; $(\zeta^{-1} f - VP_{M_0}^H f) \perp M_\infty$. Then for arbitrary $u \in D(V)$, we may write:

$$\begin{aligned} 0 &= (\zeta^{-1} f - VP_{M_0}^H f, Vu)_H = \zeta^{-1} (f, Vu)_H - (P_{M_0}^H f, u)_H \\ &= (f, \zeta V u)_H - (f, u)_H = (f, (\zeta V - E_H)u)_H. \end{aligned}$$

Therefore $f \in N_\zeta$. \square

For an arbitrary $\zeta \in \mathbb{T}$, we define an operator W_ζ by the following equality:

$$W_\zeta P_{N_0}^H f = \zeta^{-1} P_{N_\infty}^H f, \quad f \in N_\zeta, \quad (113)$$

with the domain $D(W_\zeta) = P_{N_0}^H N_\zeta$. Let us check that this definition is correct. If $g \in D(W_\zeta)$ admits two representations: $g = P_{N_0}^H f_1 = P_{N_0}^H f_2$, $f_1, f_2 \in N_\zeta$, then $P_{N_0}^H (f_1 - f_2) = 0$. By (109) this implies $P_{N_\infty}^H (f_1 - f_2) = 0$, and therefore the definition is correct. The operator W_ζ is linear, and

$$\|W_\zeta P_{N_0}^H f\|_H = \|P_{N_\infty}^H f\|_H = \|P_{N_0}^H f\|_H.$$

Thus, W_ζ is isometric. Notice that $R(W_\zeta) = P_{N_\infty}^H N_\zeta$.

Set

$$S = P_{N_0}^H|_{N_\zeta}, \quad Q = P_{N_\infty}^H|_{N_\zeta}.$$

In what follows, we suppose that ζ^{-1} is a point of the regular type of the operator V . Let us check that in this case operators S and Q are invertible.

Suppose to the contrary that there exists $f \in N_\zeta$, $f \neq 0$: $Sf = P_{N_0}^H f = 0$. Then $f = P_{M_0}^H f \neq 0$. Thus, we get by Theorem 4.1 that $f \in M_0 \cap N_\zeta = \{0\}$. We obtained a contradiction. In a similar way, suppose that there exists $g \in N_\zeta$, $g \neq 0$: $Qg = P_{N_\infty}^H g = 0$. Then $g = P_{M_\infty}^H g \neq 0$. Therefore by Theorem 4.1 we get $g \in M_\infty \cap N_\zeta = \{0\}$. We obtained a contradiction, as well.

Moreover, by Theorem 4.1 we conclude that

$$P_{N_0}^H N_\zeta = N_0; \quad P_{N_\infty}^H N_\zeta = N_\infty. \quad (114)$$

Thus, S^{-1} and Q^{-1} are closed and defined on the subspaces N_0 and N_∞ , respectively. Therefore S^{-1} and Q^{-1} are bounded. By (113) we see that $D(W_\zeta) = N_0$, $R(W_\zeta) = N_\infty$, and

$$W_\zeta = \zeta^{-1} Q S^{-1}. \quad (115)$$

The following theorem holds, see [7, Theorem 1].

Theorem 5.1 *Let V be a closed isometric operator in a Hilbert space H , and C be a linear bounded operator in H , $D(C) = N_0$, $R(C) \subseteq N_\infty$. Let $\zeta \in \mathbb{T}$, and ζ^{-1} be a point of the regular type of the operator V . The point ζ^{-1} is an eigenvalue of $V_{0;C}^+ = V \oplus C$, if and only if*

$$(C - W_\zeta)g = 0, \quad g \in N_0, \quad g \neq 0. \quad (116)$$

Proof. *Necessity.* Since ζ^{-1} is an eigenvalue of $V_{0;C}^+ = V \oplus C$, then by Lemma 5.2 there exists $f \in N_\zeta$, $f \neq 0$, such that

$$C P_{N_0}^H f = \zeta^{-1} P_{N_\infty}^H f. \quad (117)$$

Comparing the last relation with the definition of W_ζ , we see that $C P_{N_0}^H f = W_\zeta P_{N_0}^H f$. Set $g = P_{N_0}^H f = S f$. Since S is invertible, then $g \neq 0$.

Sufficiency. From (116) we get (117) with $f := S^{-1}g$. By Lemma 5.1 we see that relations (111),(112) hold. The latter, as we have seen before relation (111), is equivalent to the fact that ζ^{-1} is an eigenvalue of $V_{0;C}^+ = V \oplus C$, with the eigenvector f . \square

The following theorem is a slightly corrected version of [7, Theorem 2].

Theorem 5.2 *Let V be a closed isometric operator in a Hilbert space H , and C be a linear bounded operator in H , $D(C) = N_0$, $R(C) \subseteq N_\infty$. Let $\zeta \in \mathbb{T}$, and ζ^{-1} be a point of the regular type of the operator V . Then*

$$R\left(V_{0;C}^+ - \zeta^{-1} E_H\right) = H, \quad (118)$$

if and only if the following two conditions hold:

$$(C - W_\zeta) N_0 = N_\infty; \quad (119)$$

$$P_{M_\infty}^H M_\zeta = M_\infty. \quad (120)$$

Proof. *Necessity.* Choose an arbitrary $h \in N_\infty$. By (118) there exists $x \in H$, such that

$$\left(V_{0;C}^+ - \zeta^{-1} E_H \right) x = (V \oplus C)x - \zeta^{-1} x = h. \quad (121)$$

For an arbitrary $u \in D(V)$, we may write:

$$\begin{aligned} (x, (E_H - \zeta V)u)_H &= (x, (V^{-1} - \zeta E_H)Vu)_H = (x, ((V_{0;C}^+)^* - \zeta E_H)Vu)_H \\ &= ((V_{0;C}^+ - \zeta^{-1} E_H)x, Vu)_H = (h, Vu)_H = 0, \end{aligned}$$

and therefore $x \in N_\zeta$. Set $g = Sx \in N_0$, and using (115) write:

$$(C - W_\zeta)g = CSx - W_\zeta Sx = CSx - \zeta^{-1} Qx.$$

Since $h \in N_\infty$, we apply $P_{N_\infty}^H$ to the equality (121) to get

$$\begin{aligned} CP_{N_0}^H x - \zeta^{-1} P_{N_\infty}^H x &= h; \\ CSx - \zeta^{-1} Qx &= h. \end{aligned}$$

Therefore we obtain that

$$(C - W_\zeta)g = h,$$

and (119) holds.

Choose an arbitrary $\hat{h} \in M_\infty$. By (118) there exists $\hat{x} \in H$, such that

$$\left(V_{0;C}^+ - \zeta^{-1} E_H \right) \hat{x} = (V \oplus C)\hat{x} - \zeta^{-1} \hat{x} = \hat{h}. \quad (122)$$

The last equality is equivalent to the following two equalities obtained by applying projectors $P_{M_\infty}^H$ and $P_{N_\infty}^H$:

$$VP_{M_0}^H \hat{x} - \zeta^{-1} P_{M_\infty}^H \hat{x} = \hat{h}; \quad (123)$$

$$CP_{N_0}^H \hat{x} - \zeta^{-1} P_{N_\infty}^H \hat{x} = 0. \quad (124)$$

By Theorem 4.1 we may write:

$$\hat{x} = x_{D(V)} + x_{N_\zeta}, \quad x_{D(V)} \in D(V), \quad x_{N_\zeta} \in N_\zeta.$$

By substitution this decomposition into relation (124) we get:

$$\begin{aligned} CP_{N_0}^H x_{N_\zeta} - \zeta^{-1} P_{N_\infty}^H x_{N_\zeta} - \zeta^{-1} P_{N_\infty}^H x_{D(V)} &= 0; \\ (C - W_\zeta) P_{N_0}^H x_{N_\zeta} &= \zeta^{-1} P_{N_\infty}^H x_{D(V)}. \end{aligned} \quad (125)$$

On the other hand, by substitution of the decomposition into (123) we get:

$$\begin{aligned} Vx_{D(V)} + VP_{M_0}^H x_{N_\zeta} - \zeta^{-1} P_{M_\infty}^H x_{D(V)} - \zeta^{-1} P_{M_\infty}^H x_{N_\zeta} &= \widehat{h}; \\ Vx_{D(V)} - \zeta^{-1} P_{M_\infty}^H x_{D(V)} &= \widehat{h}, \end{aligned} \quad (126)$$

where we used Lemma 5.1. Then

$$P_{M_\infty}^H (V - \zeta^{-1} E_H) x_{D(V)} = \widehat{h},$$

and (120) follows directly.

Sufficiency. Choose an arbitrary $h \in H$, $h = h_1 + h_2$, $h_1 \in M_\infty$, $h_2 \in N_\infty$. By (119) there exists $g \in N_0$ such that

$$(C - W_\zeta) g = Cg - W_\zeta g = h_2.$$

Set $x = S^{-1}g \in N_\zeta$. Then

$$\begin{aligned} CSx - \zeta^{-1} Qx &= h_2; \\ P_{N_\infty}^H (V \oplus C) P_{N_0}^H x - \zeta^{-1} P_{N_\infty}^H x &= h_2; \\ P_{N_\infty}^H (V_{0;C}^+ x - \zeta^{-1} x) &= h_2. \end{aligned}$$

Observe that by Lemma 5.1 we may write:

$$P_{M_\infty}^H (V_{0;C}^+ x - \zeta^{-1} x) = VP_{M_0}^H x - \zeta^{-1} P_{M_\infty}^H x = 0.$$

Therefore

$$(V_{0;C}^+ - \zeta^{-1} E_H) x = h_2. \quad (127)$$

By (120) there exists $w \in M_\zeta$, such that

$$P_{M_\infty}^H w = h_1.$$

Let

$$w = (V - \zeta^{-1} E_H) \widetilde{x}_{D(V)}, \quad \widetilde{x}_{D(V)} \in D(V).$$

Then

$$V\widetilde{x}_{D(V)} - \zeta^{-1} P_{M_\infty}^H \widetilde{x}_{D(V)} = h_1. \quad (128)$$

By (119) there exists $r \in N_0$ such that

$$(C - W_\zeta) r = \zeta^{-1} P_{N_\infty}^H \tilde{x}_{D(V)}.$$

Set $\tilde{x}_{N_\zeta} := S^{-1} r \in N_\zeta$. Then

$$\begin{aligned} (C - W_\zeta) P_{N_0}^H \tilde{x}_{N_\zeta} &= \zeta^{-1} P_{N_\infty}^H \tilde{x}_{D(V)}; \\ C P_{N_0}^H \tilde{x}_{N_\zeta} - \zeta^{-1} P_{N_\infty}^H \tilde{x}_{N_\zeta} - \zeta^{-1} P_{N_\infty}^H \tilde{x}_{D(V)} &= 0. \end{aligned} \quad (129)$$

Set $\tilde{x} = \tilde{x}_{D(V)} + \tilde{x}_{N_\zeta}$. Then from (129) we get

$$C P_{N_0}^H \tilde{x} - \zeta^{-1} P_{N_\infty}^H \tilde{x} = 0. \quad (130)$$

Using (128) and Lemma 5.1 we write:

$$\begin{aligned} h_1 &= V \tilde{x}_{D(V)} - \zeta^{-1} P_{M_\infty}^H \tilde{x}_{D(V)} + V P_{M_0}^H \tilde{x}_{N_\zeta} - \zeta^{-1} P_{M_\infty}^H \tilde{x}_{N_\zeta} \\ &= V P_{M_0}^H \tilde{x} - \zeta^{-1} P_{M_\infty}^H \tilde{x}. \end{aligned} \quad (131)$$

Summing relations (130) and (131) we get

$$(V \oplus C) \tilde{x} - \zeta^{-1} \tilde{x} = \left(V_{0;C}^+ - \zeta^{-1} E_H \right) \tilde{x} = h_1. \quad (132)$$

Summing relations (127) and (132) we deduce that relation (118) holds. \square

The following theorem holds, cf. [7, Theorem 4]:

Theorem 5.3 *Let V be a closed isometric operator in a Hilbert space H , and Δ be some open arc of \mathbb{T} , such that ζ^{-1} is a point of the regular type of V , $\forall \zeta \in \Delta$. Let the following condition be satisfied:*

$$P_{M_\infty}^H M_\zeta = M_\infty, \quad \forall \zeta \in \Delta. \quad (133)$$

Let $\mathbf{R}_z = \mathbf{R}_z(V)$ be an arbitrary generalized resolvent of V , and $C(\lambda; 0) \in \mathcal{S}(N_0; N_\infty)$ corresponds to $\mathbf{R}_z(V)$ by Inin's formula (10). $\mathbf{R}_z(V)$ has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$, if and only if the following conditions are satisfied:

- 1) $C(\lambda; 0)$ has an extension to the set $\mathbb{D} \cup \Delta$ which is continuous in the uniform operator topology;
- 2) The extended $C(\lambda; 0)$ maps isometrically N_0 on the whole N_∞ , for all $\lambda \in \Delta$;

3) The operator $C(\lambda; 0) - W_\lambda$ is invertible for all $\lambda \in \Delta$, and

$$(C(\lambda; 0) - W_\lambda)N_0 = N_\infty, \quad \forall \lambda \in \Delta. \quad (134)$$

Proof. *Necessity.* Let $\mathbf{R}_z(V)$ have an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$. By Corollary 3.1 we conclude that conditions 1) and 2) are satisfied, and the operator $(E_H - \lambda V_{C(\lambda; 0); 0})^{-1} = -\frac{1}{\lambda}((V \oplus C(\lambda; 0)) - \frac{1}{\lambda}E_H)^{-1}$ exists and it is defined on the whole H , for all $\lambda \in \Delta$. By Theorem 5.1 we conclude that the operator $C(\lambda; 0) - W_\lambda$ is invertible for all $\lambda \in \Delta$. By Theorem 5.2 we obtain that relation (134) holds.

Sufficiency. Let conditions 1)-3) be satisfied. By Theorem 5.2 we get that $R((V \oplus C(\lambda; 0)) - \frac{1}{\lambda}E_H) = R(E_H - \lambda V_{C(\lambda; 0); 0}) = H$. By Theorem 5.1 we see that the operator $(V \oplus C(\lambda; 0)) - \frac{1}{\lambda}E_H$ is invertible. By Corollary 3.1 we conclude that $\mathbf{R}_z(V)$ has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$. \square

Remark 5.1 By Corollary 3.1, if $\mathbf{R}_z(V)$ has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$, then $((V \oplus C(\lambda; 0)) - \frac{1}{\lambda}E_H)^{-1}$ exists and it is bounded. Therefore points λ^{-1} , $\lambda \in \Delta$, are of the regular type for V . On the other hand, by Theorem 5.2, condition (133) holds in this case. Thus, by Proposition 3.1, condition (133) and the condition that points λ^{-1} , $\lambda \in \Delta$, are of the regular type for V , both are necessary for the existence of a spectral function \mathbf{F} of V such that $\mathbf{F}(\overline{\Delta}) = 0$. Thus, they do not imply on the generality of Theorem 5.3.

We shall obtain an analogous result in terms of the parameter $C(\lambda; z_0)$ of Inin's formula, for an arbitrary $z_0 \in \mathbb{D}$.

Theorem 5.4 Let V be a closed isometric operator in a Hilbert space H , and Δ be some open arc of \mathbb{T} , such that ζ^{-1} is a point of the regular type of V , $\forall \zeta \in \Delta$. Let $z_0 \in \mathbb{D}$ be an arbitrary fixed point, and the following condition be satisfied:

$$P_{M \frac{1}{z_0}}^H M_{\frac{z_0 + \zeta}{1 + \zeta \overline{z_0}}} = M_{\frac{1}{\overline{z_0}}}, \quad \forall \zeta \in \Delta. \quad (135)$$

Let $\mathbf{R}_z = \mathbf{R}_z(V)$ be an arbitrary generalized resolvent of V , and $C(\lambda; z_0) \in \mathcal{S}(N_{z_0}; N_{\frac{1}{\overline{z_0}}})$ corresponds to $\mathbf{R}_z(V)$ by Inin's formula (10). $\mathbf{R}_z(V)$ has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$, if and only if the following conditions are satisfied:

- 1) $C(\lambda; z_0)$ has an extension to the set $\mathbb{D} \cup \Delta$ which is continuous in the uniform operator topology;

- 2) The extended $C(\lambda; z_0)$ maps isometrically N_{z_0} on the whole $N_{\frac{1}{z_0}}$, for all $\lambda \in \Delta$;
- 3) The operator $C(\lambda; z_0) - W_{\lambda; z_0}$ is invertible for all $\lambda \in \Delta$, and

$$(C(\lambda; z_0) - W_{\lambda; z_0})N_{z_0} = N_{\frac{1}{z_0}}, \quad \forall \lambda \in \Delta. \quad (136)$$

Here $W_{\lambda; z_0}$ is defined by the following equality:

$$W_{\lambda; z_0} P_{N_{z_0}}^H f = \frac{1 - \overline{z_0} \lambda}{\lambda - z_0} P_{N_{\frac{1}{z_0}}}^H f, \quad f \in N_\lambda, \quad \lambda \in \mathbb{T}. \quad (137)$$

Proof. We first notice that in the case $z_0 = 0$ this theorem coincides with Theorem 5.3. Thus, we may assume that $z_0 \in \mathbb{D} \setminus \{0\}$.

Let $\mathbf{R}_z(V)$ have an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$. Recall that the generalized resolvent $\mathbf{R}_z(V)$ is related to the generalized resolvent $\mathbf{R}_z(V_{z_0})$ of V_{z_0} by relation (14), and that correspondence is bijective. Then $\mathbf{R}_z(V_{z_0})$ admits an analytic continuation to $\mathbb{T}_e \cup \Delta_1$, where

$$\Delta_1 = \left\{ \tilde{t} : \tilde{t} = \frac{\lambda - z_0}{1 - \overline{z_0} \lambda}, \quad \lambda \in \Delta \right\}.$$

By Proposition 2.1, points \tilde{t}^{-1} , $\tilde{t} \in \Delta_1$, are of the regular type of the operator V_{z_0} . Moreover, relation (133) written for the operator V_{z_0} with $\zeta \in \Delta_1$, coincides with relation (135). Then we may apply Theorem 5.3 for the operator V_{z_0} and the open arc Δ_1 . If we then rewrite conditions 1)-3) of that theorem in terms of $C(\lambda; z_0)$, using the bijective correspondence between $C(\lambda; z_0)$ for V , and $C(\lambda; 0)$ for V_{z_0} , we easily get conditions 1)-3) of the present theorem.

On the other hand, let conditions 1)-3) of the present theorem be satisfied. Then conditions of Theorem 5.3 for the operator V_{z_0} are satisfied. Therefore $\mathbf{R}_z(V_{z_0})$ admits an analytic continuation to $\mathbb{T}_e \cup \Delta_1$. Consequently, $\mathbf{R}_z(V)$ has an analytic continuation to the set $\mathbb{D} \cup \mathbb{D}_e \cup \Delta$. \square

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On the generalized resolvents for isometric operators with gaps.

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In this paper we obtain some slight correction and generalization of the results of Ryabtseva on the generalized resolvents for isometric operators with a gap in their spectrum. Also, analogs of some McKelvey's results and a short proof of Inin's formula for the generalized resolvents of an isometric operator are obtained.